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The division problem with voluntary participation

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Abstract The division problem consists of allocating a given amount of a homogeneous and perfectly divisible good among a group of agents with single-peaked preferences on the set of their potential shares. A rule proposes a vector of shares for each division problem. The literature has implicitly assumed that agents will find acceptable any share they are assigned to. In this article we consider the division problem when agents' participation is voluntary. Each agent has an idiosyncratic interval of acceptable shares where his preferences are single-peaked. A rule has to propose to each agent either to not participate or an acceptable share because otherwise he would opt out and this would require to reassign some of the remaining agents' shares. We study a subclass of efficient and consistent rules and characterize extensions of the uniform rule that deal explicitly with agents' voluntary participation.

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1 Introduction

The division problem consists of a set of agents that have to share an amount of an homogeneous and perfectly divisible good. Each agent has single-peaked preferences on the set of his potential shares; namely, there is an amount of the good (the peak of the agent) that is his most-preferred share and in both sides of the peak the preference is monotonic, decreasing at its right and increasing at its left. Since preferences reflect idiosyncratic characteristics of the agents, they have to be elicited by a rule that maps each division problem (a set of agents, a preference profile of declared list of single-peaked preferences, one for each agent, and the amount of the good to be allocated) into a vector of shares. But in general, the sum of the peaks will be either larger or smaller than the total amount to be allocated. A positive or negative rationing problem emerges depending on whether the sum of the peaks exceeds or falls short the fixed amount. Rules differ from each other in how this rationing problem is resolved in terms of incentives, efficiency, fairness, monotonicity, consistency, etc.

There are many examples of allocation problems that fit with this general description. For instance, a group of agents participate in an activity that requires a fixed amount of labor (measured in units of time). Agents have a maximal number of units of time to contribute and consider working as being undesirable. Suppose that labor is homogeneous and the wage is fixed. Then, strictly monotonic and quasi-concave preferences on the set of bundles of money and leisure generate single-peaked preferences on the set of potential shares where the peak is the amount of working time associated to the optimal bundle. Similarly, a group of agents join a partnership to invest in a project (an indivisible bond with a face value, for example) that requires a fixed amount of money (neither more nor less). Their risk attitudes and wealth induce single-peaked preferences on the amount to be invested. In both cases, it is required that a rule solves the rationing problem arising from a vector of peaks that do not add up the needed amount.

However, in many applications (like those described above), agents' participation cannot be compulsory. For instance, to participate agents may have to pay a fixed cost or a fee which could make smaller and larger shares—the less preferred ones given their single-peaked preferences—unacceptable. Then, each agent will have an interval of acceptable shares whose elements are preferred to opt out. Therefore, the rule cannot propose unacceptable shares to agents. In this article we study rules that solve the rationing problem when agents' participation is voluntary. We call an allocation problem of this type, a *division problem with voluntary participation* (a *problem*, for short). Now, in a problem each agent's preferences are characterized by an interval of acceptable shares where preferences are single-peaked. Only shares inside this interval are considered to be acceptable. A rule will have to propose, for each problem, a vector where each agent either does not participate or else receives an acceptable share. Consequently, the vector where no agent participates (and the good is disposed of completely) is a feasible allocation. Hence, our model applies to situations involving a perfectly divisible good that can either be disposed of completely or be allocated completely.

In a related paper Cantala (2004) considers agents' voluntary participation in the public good counterpart of the division problem with single-peaked preferences. He studies a model in which each agent can opt out from consuming the public good if its chosen level falls outside of his set of acceptable levels. An important difference between Cantala (2004) and our private good model is that when an agent opts out and does not consume the public good, the level of the public good may remain unchanged while in the private good case the shares of some of the remaining agents have to be redefined.

We are interested in rules that satisfy a set of desirable properties. First, *efficiency*. A rule is efficient if it always selects Pareto optimal allocations. Efficiency guarantees that in solving the rationing problem (either positive or negative) no amount of the good is wasted. Second, *consistency*. A rule is consistent if the proposed shares at a given problem coincide with the shares that the rule would propose at any smaller problem obtained after that a subset of agents, agreeing with the amounts the rule has assigned to them, leave the society taking with them their already assigned shares. Consistency guarantees that, in order to follow the rule's prescription at the reduced problem, the remaining agents do not have to reallocate their shares. Third, *individual rationality from equal division*. Suppose that we assign to each agent his smallest acceptable share. The rest is divided as equally as possible under the condition that no agent receives more than his largest acceptable share. A rule satisfies this property by choosing a Pareto improvement from the previous allocation.[1] Individual rationality from equal division embeds to the rule a minimal egalitarian principal only broken for two reasons. First, to keep binding the restrictions derived from the requirement that agents have to receive acceptable shares and second, to admit Pareto improvements from this egalitarian allocation. In contrast with the division problem when all shares are acceptable,[2] we show that when agents' participation is voluntary the fundamental properties of strategy-proofness, efficiency, anonymity and one-sided resource-monotonicity are incompatible. Specifically, there is no rule satisfying strategy-proofness. Besides, efficiency is also incompatible with either anonymity or one-sided resource-monotonicity. In Sect. 3.2 we give formal proofs of these incompatibilities.[3] We proceed by leaving aside incentive issues and by focusing on the class of efficient and consistent rules that are individually rational from equal division.

Before moving to the general description of our results we want to stress a fundamental attribute of rules when applied to division problems with voluntary participation. Fix a problem (a set of agents, their preferences, and the amount of the good to be allocated). A rule has to make two choices. First, it has to select a subset of agents (a coalition) among whom the good will be allocated. This coalition has to be admissible for the problem: it should be possible to allocate the total amount of the good among its members without violating their participation constraints. Second, and given this chosen coalition (if non-empty), the rule has to select (among potentially many) a

[1] See Sönmez (1994) for an analysis of rules satisfying this property in the context of division problems with compulsory participation.

[2] In this setting Sprumont (1991) characterizes the uniform rule as the unique rule satisfying efficiency, anonymity (the names of the agents do not play any role), and strategy-proofness (truth-telling is a dominant strategy in the direct revelation game induced by the rule).

[3] In contrast again, Barberà et al. (1997) shows that when agents' participation is compulsory the class of strategy-proof and efficient rules is extremely large.

particular share allotted to each of its members. When participation is compulsory rules disregard the first issue and always select the grand coalition. In this setting the uniform rule has emerged as the most appealing one.[4] At each division problem with compulsory participation the uniform rule tries to allocate the amount of the good among all agents as equally as possible, keeping the efficiency constraints binding. Hence, all agents are constrained in the same way; i.e., all agents receive either a share below their peaks (when the sum of all their ideals is larger than the total amount) or a share above their peaks (when the sum of all their ideals is smaller than the total amount).

Our results axiomatically identify three nested classes of rules. In all cases the set of axioms will single out a unique way of allocating the amount of the good among the members of an admissible chosen coalition. The classes will differ precisely on how their elements choose the admissible coalition. This unique allotting way consists of the following natural extension of the uniform rule. Fix a problem. If the empty coalition is the unique admissible one, no agent participates. Otherwise, take the chosen non-empty admissible coalition. Then the allocation of the good among its members can be described as a two-step procedure. First, assign to each agent in the coalition his smallest acceptable share. The remainder is assigned by adding uniformly the same amount to every agent in the coalition. If the sum of the peaks exceeds the amount to be allocated then the rule stops adding to those agents whose peak is reached, and keeps adding uniformly to the rest. Observe that in this case the remainder will eventually be exhausted before all peaks are reached. If the sum of the peaks is smaller than the amount to be allocated then the rule also keeps adding uniformly to all agents, and stops adding only to those agents whose largest acceptable share is reached, and keeps adding uniformly to the rest. Observe now that since the coalition was admissible the remainder will eventually be exhausted before reaching all largest acceptable shares. We call any rule satisfying this allotment procedure an *extended uniform rule*. There are many because at many problems there are many admissible coalitions. Hence, extended uniform rules differ only on the choice of the subset of agents among whom the amount of the good is allocated.

Theorem 1 characterizes the class of efficient, consistent and individually rational from equal division rules as the subset of extended uniform rules that select the admissible coalition by choosing coherently the full set of agents whenever it is possible. Theorem 2 characterizes the subclass of rules that, in addition to the previous properties, satisfy an independence of irrelevant alternatives like property (that we call *independence of irrelevant coalitions*). This class consists of the subset of extended uniform rules that at each problem choose the admissible coalition by maximizing a given monotonic order on the set of all finite coalitions. Theorem 3 characterizes the smaller subclass of rules that in addition to efficiency, consistency, and individually rationality form equal division also satisfy *order priority* with respect to a given order among individual agents. This class consists of the subset of extended uniform rules that at each problem choose the admissible coalition by selecting lexicographically

[4] See Ching (1992, 1994), Schummer and Thomson (1997), Sönmez (1994), Sprumont (1991), Thomson (1994a, 1995, 1997), and Weymark (1999) for alternative characterizations of the uniform rule in the division problem. In the surveys on strategy-proofness of Barberà (1996, 2001, 2010), Jackson (2001) and Sprumont (1995) the division problem and the uniform rule plays a prominent role.

according to the given order. We also show that in all three characterizations the axioms are independent.

The article is organized as follows. In Sect. 2 we describe the model. In Sect. 3 we define several properties that a rule may satisfy and show some basic incompatibilities among them. In Sect. 4 we define extended uniform rules. In Sect. 5 we present the main results of the article. In Sect. 6 we conclude with a discussion and some final remarks. Three appendices at the end of the article collect the proofs of the three theorems.

2 The model

Let $t > 0$ be a fixed amount of a homogeneous and perfectly *divisible good*. A finite set of *agents* is considering the possibility of dividing t among a subset of them, to be determined according to their preferences. Since we will be considering situations where the amount of the good t and the finite set of agents may vary, let $\mathbb{N}$ be the set of positive integers and let $\mathcal{N}$ be the family of all non-empty and finite subsets of $\mathbb{N}$. The set of agents is then $N \in \mathcal{N}$ with cardinality n. In contrast with Sprumont (1991), we consider situations where each agent has the right to opt out of the division problem. A feasible allocation is that no agent participates and the good is not divided at all. Observe that we are considering a perfectly divisible good that can either be disposed of completely or be allocated completely. We denote by NP the alternative of not participating. Thus, and since each agent i cannot be forced to receive an unacceptable share of the good, his preferences $\succeq_i$ are defined on the set $\{\text{NP}\} \cup [l_i, u_i]$, where $[l_i, u_i] \subseteq [0, +\infty]$ is agent i's *interval of acceptable shares*. We assume that $\succeq_i$ is a complete, reflexive, and transitive binary relation on $\{\text{NP}\} \cup [l_i, u_i]$. Given $\succeq_i$ let $\succ_i$ be the antisymmetric binary relation induced by $\succeq_i$ (i.e., for all $x_i, y_i \in \{\text{NP}\} \cup [l_i, u_i]$, $x_i \succ_i y_i$ if and only if $y_i \succeq x_i$ does not hold) and let $\sim_i$ be the indifference relation induced by $\succeq_i$ (i.e., for all $x_i, y_i \in \{\text{NP}\} \cup [l_i, u_i]$, $x_i \sim_i y_i$ if and only if $x_i \succeq_i y_i$ and $y_i \succeq x_i$). We will also assume that $\succeq_i$ is single-peaked on $[l_i, u_i]$ and we will denote by $p_i \in [l_i, u_i]$ agent i's *peak*. Formally, agent i's preferences $\succeq_i$ is a complete preorder on the set $\{\text{NP}\} \cup [l_i, u_i]$ that satisfies the following additional properties:

(P.1) there exists $p_i \in [l_i, u_i]$ such that $p_i \succ_i x_i$ for all $x_i \in [l_i, u_i] \backslash \{p_i\}$;
(P.2) $x_i \succ_i y_i$ for any pair of shares $x_i, y_i \in [l_i, u_i]$ such that either $y_i < x_i \leq p_i$ or $p_i \leq x_i < y_i$;
(P.3) $x_i \succ_i \text{NP}$ for all $x_i \in (l_i, u_i)$;
(P.4) if $0 < u_i < +\infty$ then $l_i \sim_i u_i$; and
(P.5) if $u_i = +\infty$ then $l_i \prec_i x_i$ for all $x_i > l_i$.

A motivation for this kind of preferences is the following. Let agent i's preferences be single-peaked and continuous on $[0, +\infty)$. Now we add the option NP to $[0, +\infty)$ and there exist l_i, u_i with x_i is strictly preferred to NP for all $x_i \in (l_i, u_i)$. The properties (P.1)–(P.5) are readily verified.

Observe that agent i's preferences are independent of t and are defined on the set $\{\text{NP}\} \cup [l_i, u_i]$, which will also be considered private information when we define rules on the set of profiles. Conditions (P.1) and (P.2) state that $\succeq_i$ is single-peaked on $[l_i, u_i]$. Condition (P.3) follows from single-peakedness on $[l_i, u_i]$ and the desirability of acceptable shares. Conditions (P.4) and (P.5) allow to interpret the interval of acceptable shares $[l_i, u_i]$ as a truncation of an original single-peaked preference on $[0, +\infty)$, where the truncation arises from the fact that agents may opt out (as in Cantala 2004). In particular, (P.4) and (P.5) help to give sense to this truncation interpretation. Nevertheless, all our results also hold in the domain of preferences satisfying (P.1), (P.2), and (P.3).[5] Note that the domain of preferences satisfying conditions (P.1)–(P.5) is large because we are admitting several possibilities. First, that agent i only has one acceptable share (i.e., $l_i = p_i = u_i$).[6] Second, that $l_i > 0$ to reflect the case where to receive a positive share agents may have to incur with a (potentially small) cost; for example, the cost of writing a contract specifying the share of an indivisible bond or a lottery ticket that each agent is entailed to. Third, that agent i perceives NP as receiving indeed the 0 share (in which case NP $\sim_i l_i$ if $l_i = 0$). Fourth, that $l_i \succ_i$ NP and $u_i \succ_i$ NP to admit the case that opting out were (perhaps lexicographically) worse for the agent than staying and getting either l_i or u_i. Although we do not require any utility representation of agents' preferences, Fig. 1 illustrates three possible preferences (represented by utility functions) satisfying properties (P.1)–(P.5).

From a preference $\succeq_i$ of agent i we can associate a unique triple (l_i, p_i, u_i). There are many preferences of agent i with the same (l_i, p_i, u_i); however, they differ only on how two shares on different sides of p_i are ordered while all of them coincide on the ordering on the shares on each of the sides of p_i. A *profile* $\succeq_N = (\succeq_i)_{i \in N}$ is an n-tuple of preferences satisfying properties (P.1), (P.2), (P.3), (P.4), and (P.5) above. Given a profile $\succeq_N$ and agent i's preferences $\succeq_i'$ we denote by $(\succeq_i', \succeq_{N\setminus\{i\}})$ the profile where $\succeq_i$ has been replaced by $\succeq_i'$ and all other agents have the same preferences. When no confusion arises we denote the profile $\succeq_N$ by $\succeq$.

A division problem with voluntary participation (a *problem* for short) is a triple $(N, \succeq, t)$ where N is the set of agents, $\succeq$ is a profile and t is the amount of the good to be divided. Let $\mathcal{P}$ be the set of all problems. A situation where for all agents their participation is compulsory and preferences are single-peaked on $[0, +\infty)$ is known as the *division problem* (see Ching and Serizawa 1998).

Let $\succeq$ be a profile. Define $X(\succeq) \equiv \prod_{i \in N}(\{\text{NP}\} \cup [l_i, u_i])$. Observe that the set $X(\succeq)$ depends on the profile $\succeq$ since, for each agent $i \in N$, the set $\{\text{NP}\} \cup [l_i, u_i]$ is where i's preferences are defined. For each $x \in X(\succeq)$ denote the subset of agents that participate (and receive an acceptable share) by $S(x) = \{i \in N \mid x_i \in [l_i, u_i]\}$. Then, the set of *feasible allocations* of problem $(N, \succeq, t)$ is

$$\text{FA}(N, \succeq, t) = \left\{ x \in X(\succeq) \mid \text{if } S(x) \neq \varnothing \text{ then } \sum_{j \in S(x)} x_j = t \right\}.$$

Again, free disposal of the good is binary in the sense that either t is completely divided or it is not divided at all. Consequently, the set of feasible allocations is never empty since the allocation $x = (\text{NP}, \ldots, \text{NP}) \in X(\succeq)$ is always feasible ($S(x) = \varnothing$).

[5] In this larger domain we could admit preferences $\succeq_i$ with the property that $p_i \succ_i l_i = 0 \succ_i u_i$ or $p_i = l_i = 0 \succ_i u_i$.

[6] The use of these degenerated preferences simplifies some proofs although our results would still hold if we require that $l_i < u_i$ (see the last section for a comment on this issue).

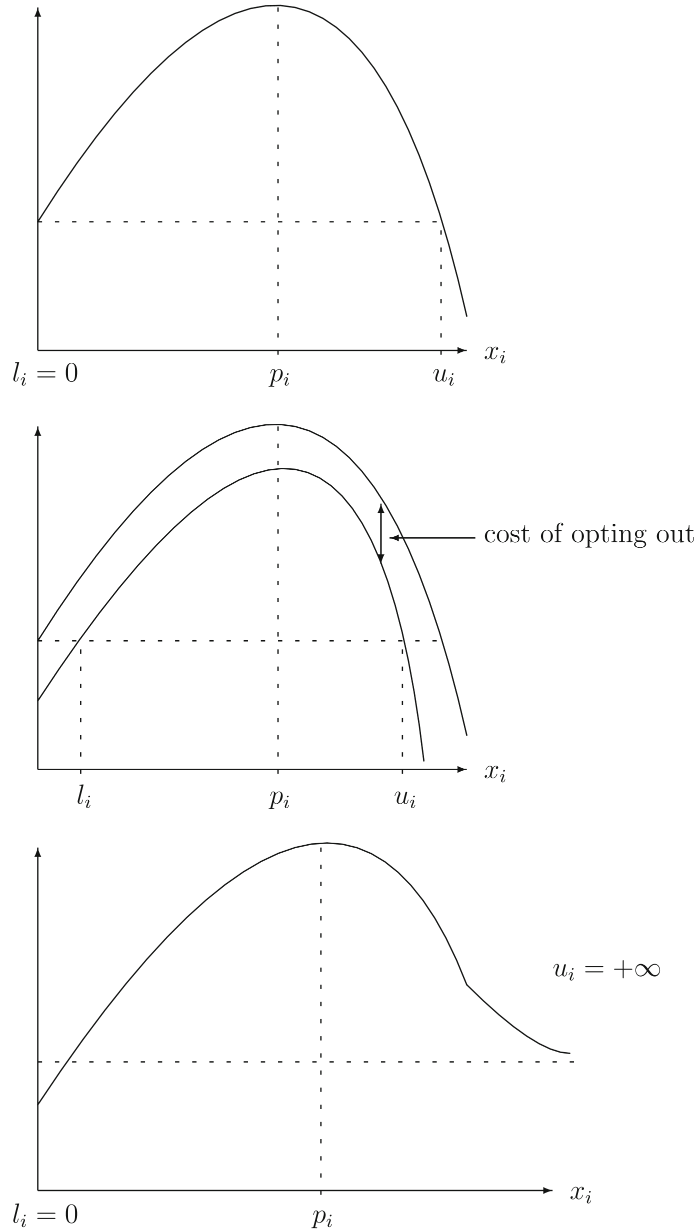

Fig. 1

Moreover, there are problems for which (NP, ..., NP) is the unique feasible allocation; for instance the problem $(N, \succeq, t)$ where $N = \{1, 2\}$, $t = 10$, and $\succeq_1$ and $\succeq_2$ are characterized by $(l_1, p_1, u_1) = (l_2, p_2, u_2) = (1, 2, 3)$.

A coalition $S \subseteq N$ is *admissible* (at profile $\succeq$ and amount t) if it is either empty or it is possible to divide t among the agents in S according to their preferences; namely, coalition $S \neq \varnothing$ is *admissible* at $(N, \succeq, t)$ if there exists $x \in \mathrm{FA}(N, \succeq, t)$ such that $S(x) = S$. It is obvious that $S \neq \varnothing$ is admissible if and only if $\sum_{j \in S} l_j \leq t \leq \sum_{j \in S} u_j$. We denote by $\mathrm{AC}(N, \succeq, t)$ the set of all admissible coalitions at $(N, \succeq, t)$. Namely,

$$\mathrm{AC}(N, \succeq, t) = \{S \subseteq N \mid S \text{ is admissible at } (N, \succeq, t)\}.$$

Observe that $\mathrm{AC}(N, \succeq, t)$ is never empty because it always contains the empty coalition.

A *rule* f assigns to each problem in $\mathcal{P}$ a feasible allocation in such a way that f selects $(\mathrm{NP}, \ldots, NP)$ at $(N, \succeq, t)$ if and only if the empty coalition is the unique admissible coalition at $(N, \succeq, t)$; that is, $f(N, \succeq, t) \in \mathrm{FA}(N, \succeq, t)$ for all $(N, \succeq, t) \in \mathcal{P}$ and $f(N, \succeq, t) = (\mathrm{NP}, \ldots, \mathrm{NP})$ if and only if $\mathrm{AC}(N, \succeq, t) = \{\varnothing\}$.[7] Hence, a rule f can be seen as a systematic way of assigning to each $(N, \succeq, t) \in \mathcal{P}$ the two different aspects of the solution of the problem. First, the admissible coalition $S \in \mathrm{AC}(N, \succeq, t)$. If $S \neq \varnothing$ we denote it by

$$c^f(N, \succeq, t) = \{i \in N \mid f_i(N, \succeq, t) \in [l_i, u_i]\}.$$

Obviously, if $i \notin c^f(N, \succeq, t)$ then $f_i(N, \succeq, t) = \mathrm{NP}$. Second, how the amount t is divided among the members of $c^f(N, \succeq, t)$; i.e.,

$$\sum_{j \in c^f(N, \succeq, t)} f_j(N, \succeq, t) = t.$$

We will later see that to identify rules satisfying appealing properties we may have some freedom when choosing one among the set of admissible coalitions while the properties will determine a unique way of dividing the amount of the good.

3 Properties of rules

3.1 Definitions

In this section, we define several properties that a rule may satisfy.

Rules require each agent to report a preference. A rule is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully; namely, it induces truth-telling as a dominant strategy in the direct revelation game generated by the rule. Given a problem $(N, \succeq_N, t)$ we say that agent $i \in N$ *manipulates* f *at profile* $\succeq_N$ *via* $\succeq'_i$ if $f_i(N, (\succeq'_i, \succeq_{N \setminus \{i\}}), t) \succ_i f_i(N, \succeq_N, t)$.

(Strategy- proofness) A rule f is *strategy-proof* if no agent can manipulate it at any profile.

[7] Note that we are requiring that, at any problem $(N, \succeq, t)$ for which there exists a non-empty coalition $S \in \mathrm{AC}(N, \succeq, t)$, $(\mathrm{NP}, \ldots, \mathrm{NP})$ is not selected by f. Since we will only be interested on efficient rules, this requirement will become relevant only when t is equal to the sum of left or upper bounds of all non-empty and admissible coalitions. To require that only in this case (i.e., when efficiency does not discriminate between the empty and the non-empty admissible coalitions) the rule selects a non-empty coalition is for technical reasons since it allows the use of easier arguments in some of the proofs.

A rule is anonymous if it only depends on the characteristics of the profile and not on the name of the agents having the corresponding preference; that is, it is invariant with respect to the index given to the agents. Let $N \in \mathcal{N}$ be a set of agents, $\tau_N : N \to N$ be a one-to-one mapping, and $\succeq_N$ be a profile. Define the profile $\tau_N(\succeq_N) \equiv (\succeq_{\tau_N(i)})_{i \in N}$.
(ANONYMITY) A rule f is *anonymous* if for any $N \in \mathcal{N}$, any one-to-one mapping $\tau_N : N \to N$ and any problem $(N, \succeq_N, t)$, $f_i(N, \succeq_N, t) = f_{\tau_N(i)}(N, \tau_N(\succeq_N), t)$ for all $i \in N$.

A rule is efficient if it always selects a Pareto optimal allocation.
(EFFICIENCY) A rule f is *efficient* if for each problem $(N, \succeq, t)$ there is no feasible allocation $(y_j)_{j \in N} \in \mathrm{FA}(N, \succeq, t)$ with the property that $y_i \succeq_i f_i(N, \succeq, t)$ for all $i \in N$ and $y_j \succ_j f_j(N, \succeq, t)$ for some $j \in N$.

Another property a rule may satisfy is related to its behavior when the amount t to be shared changes. One-sided resource-monotonicity only imposes conditions on the rule whenever the change of the amount to be shared does not change the sign of the rationing problem: if the good is scarce, an increase of the amount to be shared should make all agents better off and if the good is too abundant, a decrease of the amount to be shared should make all agents better off.[8]
(ONE- SIDED RESOURCE- MONOTONICITY) A rule satisfies *one-sided resource-monotonicity* if for all two problems $(N, \succeq, t), (N, \succeq, t') \in \mathcal{P}$ with the property that either $t \leq t' \leq \sum_{i \in N} p_i$ or $\sum_{i \in N} p_i \leq t' \leq t$ then $f_i(N, u, \succeq, t') \succeq_i f_i(N, u, \succeq, t)$ for all $i \in N$.

A rule is consistent if the following requirement holds. Apply the rule to a given problem and assume that a subset of agents leave with their corresponding shares. Consider the new problem formed by the set of agents that remain with the same preferences that they had in the original problem and the total amount of the good minus the sum of the shares received by the subset of agents that already left. Then the rule does not require to reallocate the shares of the remaining agents.
(CONSISTENCY) A rule f is *consistent* if for each problem $(N, \succeq_N, t)$, each non-empty subset of agents $S \subset N$, and each $i \in S$,

$$f_i(N, \succeq_N, t) = f_i\left(S, \succeq_S, t - \sum_{j \in c^f(N, \succeq_N, t) \setminus S} f_j(N, \succeq_N, t)\right).$$

For the division problem with compulsory participation Sönmez (1994) proposed the principle of individual rationality from equal division. A rule f is individually rational from equal division if all agents receive a share that is at least as good as the equal division share; namely, for each division problem $(N, \succeq, t)$,

$$f_i(N, \succeq, t) \succeq_i \frac{t}{n}$$

for all $i \in N$. In a division problem equal division is always feasible but often is not efficient. Precisely, this principle tries to make compatible equal division with efficiency by allowing for Pareto improvements from the equal division share. Observe

[8] See Thomson (1994b) and Sönmez (1994) for a discussion of one-sided resource monotonicity and axiomatic characterizations using this property.

that in our setting the allocation $(\frac{t}{n}, \ldots, \frac{t}{n})$ may not be feasible and/or there may not even exist a vector of shares at which all agents are better off than at equal division. Thus, when agents' participation is voluntary, this property is too strong (no rule satisfies it) and it cannot be applied directly. However, and since we think that its content is appealing we suggest to use the same principle as follows. Assume that in the problem $(N, \succeq, t)$ the coalition N is admissible. Preliminarily, assign to each agent i the amount l_i (which is possible since N is admissible). The remaining amount $t-\sum_{j\in N} l_j$ has still to be allocated, but again, by feasibility, each agent i must receive overall at most u_i. Then, allocate the remaining amount $t-\sum_{j\in N} l_j$ as equally as possible, but making sure that no agent i receives additionally more than $u_i - l_i$. Each agent must receive a share at least as good as the previous allocation. Formally,

(Individual Rationality from Equal Division) A rule f is *individual rational from equal division* if for each problem $(N, \succeq, t)$ for which N is an admissible coalition,

$$f_i(N, \succeq, t) \succeq_i l_i + \min\{\alpha, u_i - l_i\}$$

for all $i \in N$, where $\alpha \in \mathbb{R}$ satisfies $\sum_{j\in N} \min\{\alpha, u_j - l_j\} = t - \sum_{j\in N} l_j$.[9]

The next two properties refer explicitly on how the rule chooses the admissible coalition.

A rule satisfies independence of irrelevant coalitions if the following requirement holds. Consider two problems where the set of admissible coalitions of the first one is contained in the set of admissible coalitions of the second one. Assume that the coalition chosen by the rule in the second problem is admissible for the first one. Then, the rule chooses the same coalition in the two problems. As in many other settings, this principle adopts a revealed preference point of view: if something is chosen in a set (and thus, it is revealed as being as preferred to all other alternatives in that set) and the set becomes smaller but still contains what has been chosen, the new choice should not change.

(Independence of irrelevant coalitions) A rule f satisfies *independence of irrelevant coalitions* if for any two problems $(N, \succeq, t)$ and $(N', \succeq', t')$ such that $\mathrm{AC}(N', \succeq', t') \subset \mathrm{AC}(N, \succeq, t)$ and $c^f(N, \succeq, t) \in \mathrm{AC}(N', \succeq', t')$ then,

$$c^f(N', \succeq', t') = c^f(N, \succeq, t).$$

An order σ is a one-to-one mapping $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$. A rule satisfies order priority with respect to σ if agent i has more rights to be in the coalition sharing t than any agent that goes after him according to σ.[10] Namely,

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9 Note that in the division problem with compulsory participation our version of the principle says that $f_i(N, \succeq, t) \succeq_i \frac{t}{n}$ for all $i \in N$. Observe that in the voluntary participation context there are other alternative and natural ways of formalizing the idea of individual rationality from equal division. In Sect. 6 we describe the one that uses as reference allotment the one obtained by starting at the vector of upper bounds decreases uniformly agents' shares as long as lower bounds are not reached.

10 Priority rules appear in many settings where to treat agents equally is unfeasible. This very asymmetric rules are still interesting because they can be used to achieve ex-ante symmetry by choosing random mechanisms whose supports are priority rules.

(ORDER PRIORITY) A rule f satisfies *order priority with respect to* σ if for each problem $(N, \succeq, t)$ such that $i \notin c^f(N, \succeq, t)$ and $c^f(N, \succeq, t) \cap \{j \in N \mid \sigma(j) > \sigma(i)\} \neq \varnothing$ then, there is no admissible coalition containing $(\{i\} \cup \{j \in N \mid \sigma(j) < \sigma(i)\}) \cap c^f(N, \succeq, t)$.

Remark 1 Let σ be an order and assume that f satisfies order priority with respect to σ. Then, f satisfies independence of irrelevant coalitions.

3.2 Some basic incompatibilities

Proposition 1 below shows that strategy-proofness is a very strong requirement when agents' participation is voluntary. The reason is that the rule has to depend not only on the agents' peaks but also on their intervals of acceptable shares; this makes it too vulnerable to manipulation. Thus, there is no strategy-proof rule. Furthermore, Proposition 1 also states that efficiency is incompatible with either anonymity or one-sided resource-monotonicity.

Proposition 1
(1.1) *There is no strategy-proof rule.*
(1.2) *There is no efficient and anonymous rule.*
(1.3) *There is no efficient and one-sided resource-monotonic rule.*

Proof To prove (1.1) let $N = \{1, 2\}$ be the set of agents, $t = 10$ and consider any profile $\succeq = (\succeq_1, \succeq_2)$ with $(l_1, p_1, u_1) = (l_2, p_2, u_2) = (4, 6, 9)$. Since the only admissible coalition is N, N is chosen. Thus, either $f_1(N, \succeq, t) < 6$ or $f_2(N, \succeq, t) < 6$. Assume, for instance, that $f_1(N, \succeq, t) < 6$. Now let agent 1 report any preference $\succeq_1'$ with $(l_1', p_1', u_1') = (6, 6, 6)$. In the problem $(N, (\succeq_1', \succeq_2), t)$, N is the only admissible coalition and hence N is chosen. Since the only feasible allocation is $(6, 4)$, $f_1(N, (\succeq_1', \succeq_2), t) = 6 \succ_1 f_1(N, \succeq, t)$, which means that f is not strategy-proof.

To prove (1.2), let $N = \{1, 2\}$ be the set of agents, $t = 10$ and consider any profile $\succeq = (\succeq_1, \succeq_2)$ with $(l_i, p_i, u_i) = (8, 9, 10)$ for $i = 1, 2$. Since $\mathrm{AC}(\{1, 2\}, \succeq, 10) \neq \{\varnothing\}$, $f(\{1, 2\}, \succeq, 10) \neq (\mathrm{NP}, \mathrm{NP})$. Hence, either $f(\{1, 2\}, \succeq, 10) = (\mathrm{NP}, 10)$ or $f(\{1, 2\}, \succeq, 10) = (10, \mathrm{NP})$, which means that f is not anonymous.

To prove (1.3), let $(N, \succeq, t)$ be such that $N = \{1, 2, 3\}$, $(l_i, p_i, u_i) = (5, 6, 8)$ for all $i \in N$, and $t = 12$. By efficiency, two agents receive 6 and the other agent receives 0. Assume without loss of generality that $f(N, \succeq, t) = (6, 6, 0)$. Let $(N, \succeq, t')$ be such that $t' = 15$. By efficiency, two general cases are possible. First, agent i_1 receives x, agent i_2 receives $15 - x$, and agent i_3 receives 0, in which case f violates one-sided resource-monotonicity because agent 1 or agent 2 receives a share that is strictly worst than 6. Second, each agent receives 5, in which case f violates one-sided resource-monotonicity because agents 1 and 2 are strictly worst off. □

4 The uniform rule and some of its extensions

The uniform rule (Sprumont 1991) has played a central role in the division problem with compulsory participation because it is the unique rule satisfying different sets of desirable properties. For instance, Sprumont (1991) shows that the uniform rule is the unique rule satisfying strategy-proofness, efficiency and anonymity.

The *uniform rule* U is defined as follows: for each division problem $(N, \succeq, t)$ and for each $i \in N$,

$$U_i(N, \succeq, t) = \begin{cases} \min\{\beta, p_i\} & \text{if } \sum_{j \in N} p_j \geq t \\ \max\{\beta, p_i\} & \text{if } \sum_{j \in N} p_j < t, \end{cases}$$

where β is the unique number satisfying $\sum_{j \in N} U_j(N, \succeq, t) = t$. Namely, U tries to allocate the good as equally as possible, keeping the efficient constraints binding: if $\sum_{j \in N} p_j \geq t$ then $U_i(N, \succeq, t) \leq p_i$ for all $i \in N$, and if $\sum_{j \in N} p_j < t$ then $U_i(N, \succeq, t) \geq p_i$ for all $i \in N$.

Observe that when applied to division problems with voluntary participation U is not a rule since at some problems it may choose non-feasible allocations. In the rest of this section we extend the uniform rule to our environment. We do it in two steps. First, we extend the uniform rule only to the subclass of problems where the grand coalition is admissible and the lower bounds of agents' intervals of acceptable shares are equal to zero. Let $(N, \succeq, t)$ be a problem with the properties that $N \in \mathrm{AC}(N, \succeq, t)$ and $l_i = 0$ for all $i \in N$. Then, define F at $(N, \succeq, t)$ as follows: for all $i \in N$,

$$F_i(N, \succeq, t) = \begin{cases} \min\{\beta, p_i\} & \text{if } \sum_{j \in N} p_j \geq t \\ \min\{\max\{\beta, p_i\}, u_i\} & \text{if } \sum_{j \in N} p_j < t, \end{cases}$$

where β is the unique number satisfying $\sum_{j \in N} F_j(N, \succeq, t) = t$. Notice that when $\sum_{j \in N} p_j \geq t$ (the upper bounds of the participation intervals do not play any role) F coincides with the uniform rule. When $\sum_{j \in N} p_j < t$ some of the upper bounds may be binding, so F makes sure that, for all $i \in N$, $\max\{\beta, p_i\}$ is never larger than u_i.

But F is not a rule itself because it only applies to a subclass of problems. To define a rule f that extends the egalitarian principle behind the uniform rule (by keeping the bounds imposed by efficiency and voluntary participation), select for each problem $(N, \succeq, t)$ an admissible coalition. If the empty set is the unique admissible coalition at $(N, \succeq, t)$, set $f_i(N, \succeq, t) = \mathrm{NP}$ for all $i \in N$. Otherwise, let $c^f(N, \succeq, t)$ be the (non-empty) admissible coalition (chosen by f) among whom t is allocated in two steps.[11] First, preliminarily assign to each agent in the chosen coalition $c^f(N, \succeq, t)$ the lower bound of his interval of acceptable shares, and then apply the rule F to the adjusted problem where the set of agents is $c^f(N, \succeq, t)$ and their preferences are 0-normalized. Formally, let $(N, \succeq, t)$ be a problem and let S be one of its non-empty admissible coalitions. The adjusted problem $(S, (\succeq_j^l)_{j \in S}, t - \sum_{j \in S} l_j)$ is the problem where S is the set of agents, and for each $i \in S$, $\succeq_i^l$ is characterized by the triple

[11] Remember that for a given problem there may be many admissible coalitions; hence, to fully describe the rule f we will have to specify how $c^f(N, \succeq, t)$ is chosen by f. But we will deal with this selection later on.

$(0, p_i - l_i, u_i - l_i)$ and given any pair $x_i, y_i \in [0, u_i - l_i]$, $x_i \succeq_i^l y_i$ if and only if $x_i + l_i \succeq_i y_i + l_i$; i.e., $\succeq_i^l$ translates $\succeq_i$ to the left by subtracting l_i.[12]

(EXTENDED UNIFORM RULE) We say that f is an *extended uniform rule* if for all $(N, \succeq, t) \in \mathcal{P}$ and all $i \in N$, $f_i(N, \succeq, t) = \text{NP}$ whenever $\text{AC}(N, \succeq, t) = \{\emptyset\}$ and otherwise,

$$f_i(N, \succeq, t) = \begin{cases} l_i + F_i(c^f(N, \succeq, t), (\succeq_j^l)_{j \in c^f(N,\succeq,t)}, t - \sum_{j \in c^f(N,\succeq,t)} l_j) & \text{if } i \in c^f(N, \succeq, t) \\ \text{NP} & \text{if } i \notin c^f(N, \succeq, t), \end{cases}$$

where $c^f(N, \succeq, t) \in \text{AC}(N, \succeq, t)$ and $c^f(N, \succeq, t) \neq \emptyset$.

Observe again that there are many problems with more than one admissible coalition and hence, there are many extended uniform rules. We exhibit an example of a rule in this family by describing a procedure to select, for each problem, an admissible coalition. This procedure is based on the idea of selecting the admissible coalition by given priority to agents according to a fixed order σ.

To roughly describe the procedure assume momentarily that $N = \{1, \ldots, n\}$ and $\sigma(i) = i$ for all $i \in N$. If the empty coalition is the unique admissible coalition at $(N, \succeq, t)$ then, choose the empty coalition and the rule assigns NP to each agent. If there are non-empty admissible coalitions at $(N, \succeq, t)$ preselect first those coalitions containing agent 1; if there are several, keep only those containing also agent 2, and so on. If there are no admissible coalitions containing agent 1, preselect those coalitions containing agent 2; if there are several, keep only those containing also agent 3, and so on.

The formal definition is recursive and depends on the one-to-one mapping $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$. Given $N \in \mathcal{N}$ and $1 \leq k \leq n$ let (abusing a bit the notation) $\sigma^{-1}(k) \equiv i$ be the agent in N such that $|\{j \in N \mid \sigma(j) \leq \sigma(i)\}| = k$; namely, $\sigma^{-1}(1)$ is the agent that goes first according to the order σ, and in general, for $1 \leq k \leq n$, $\sigma^{-1}(k)$ is the agent that has exactly $k - 1$ agents before him according to σ. Thus, given σ, we define the extended uniform rule F^σ as follows. If $\text{AC}(N, \succeq, t) = \{\emptyset\}$ then set $F_i^\sigma(N, \succeq, t) = \text{NP}$ for all $i \in N$. Assume now that the set of admissible coalitions $\text{AC}(N, \succeq, t)$ for problem $(N, \succeq, t)$ contains at least one non-empty coalition.

- Stage 0 (initialization): Given $\text{AC}(N, \succeq, t)$, set $X^0 \equiv \text{AC}(N, \succeq, t)$ and go to Stage 1.
- Stage 1 (definition of X^1): Given X^0, the output of Stage 0.
  1. If for each $S \in X^0$, $\sigma^{-1}(1) \notin S$ then, set $X^1 \equiv X^0$ and go to Stage 2.
  2. If there exists $S \in X^0$ such that $\sigma^{-1}(1) \in S$ then, set $X^1 \equiv \{S \in X^0 \mid \sigma^{-1}(1) \in S\}$ and go to Stage 2.
- Stage k (definition of X^k): Given X^{k-1}, the output of Stage $k - 1$.
  1. If for each $S \in X^{k-1}$, $\sigma^{-1}(k) \notin S$ then, set $X^k \equiv X^{k-1}$ and go to Stage $k+1$.
  2. If there exists $S \in X^{k-1}$ such that $\sigma^{-1}(k) \in S$ then, set $X^k \equiv \{S \in X^{k-1} \mid \sigma^{-1}(k) \in S\}$ and go to Stage $k + 1$.

[12] See Herrero and Villar (2000) for general translations of preferences used to define the axiom of Agenda-independence.

The procedure stops at Stage n with $X^n \equiv X^n(N, \succeq, t)$ having a unique coalition. Observe that $X^n(N, \succeq, t) \in \mathrm{AC}(N, \succeq, t)$. Then, the *$\sigma$-extended uniform rule F^σ* is the extended uniform rule such that, for each $(N, \succeq, t) \in \mathcal{P}$, $F_i^\sigma(N, \succeq, t) = \mathrm{NP}$ for all $i \in N$ whenever $\mathrm{AC}(N, \succeq, t) = \{\varnothing\}$ and $c^{F^\sigma}(N, \succeq, t) = X^n(N, \succeq, t)$ otherwise.

5 Results

We are now ready to describe and state the main results of the article. They axiomatically identify three nested subclasses of extended uniform rules. All of them use the same principle to allocate the amount of the good (the same one used by the uniform rule for division problems with compulsory participation) but differ on how to select the admissible coalition. The larger class imposes only two restrictions on the choice of the admissible coalition. First, it chooses the full set of agents whenever it is admissible. Second, it chooses the coalition coherently. The three axioms characterizing this class are efficiency, consistency and individual rationality from equal division. The intermediate class consists of those extended uniform rules that choose the admissible coalition according to a priority relation among all groups of agents that comes from a given monotonic order. This priority ordering on $\mathcal{N}$ has to be monotonic in a double sense. First, adding an agent to a given set gives priority to the larger set. Second, if a set S has priority over a set T then the priority is maintained after adding a player $i \notin S \cup T$ to both sets. This class is identified by the same axioms characterizing the larger class plus the property of independence of irrelevant coalitions. Finally, the smaller class consists of those extended uniform rules that choose the admissible coalition according to an order σ on $\mathbb{N}$ that gives priority directly to agents; namely, it is the class of all σ-extended uniform rules that have been defined in the previous section. This class consists of all efficient, consistent, and individually rational from equal division rules that satisfy order priority with respect to some σ. We now turn to the formal statements of the three results.

Theorem 1 characterizes all efficient, consistent, and individually rational from equal division rules as a subclass of extended uniform rules.

Theorem 1 *Let f be a rule. Then, f is efficient, consistent, and individually rational from equal division if and only if f is an extended uniform rule with the properties that, for all $(N, \succeq, t) \in \mathcal{P}$,*

(1.a) $c^f(N, \succeq, t) = N$ *when N is an admissible coalition at $(N, \succeq, t)$.*
(1.b) $c^f(S, \succeq_S, t - \sum_{i \in c^f(N,\succeq,t)\setminus S} f_i(N, \succeq, t)) = c^f(N, \succeq, t) \cap S$ *for each $S \subset N$.*

Proof See Appendix 1. □

There are many extended uniform rules that are inefficient, inconsistent and do not satisfy individual rationality from equal division because the choice of the admissible coalition may be extremely arbitrary. Conditions (1.a) and (1.b) in Theorem 1 precisely select those extended uniform rules that satisfy the three desirable conditions. Observe that consistency of a rule is an invariance property about the shares received by the remaining agents after a subset of agents leave the problem with their allotment. In contrast, condition (1.b) in Theorem 1 is a sort of consistency requirement on c^f that does not impose any constraint on agents' shares. In particular, (1.b) says that, for any problem $(N, \succeq, t)$, if $S \subset c^f(N, \succeq, t)$ then $c^f(S, \succeq_S, t - \sum_{i \in c^f(N,\succeq,t) \setminus S} f_i(N, \succeq, t)) = S$.

Theorem 2 characterizes all efficient, consistent, and individually rational from equal division rules that satisfy independence of irrelevant coalitions as the subclass of extended uniform rules with the property that they choose the admissible coalition according to a monotonic order given directly to coalitions (which is not necessarily induced by a unique order of agents). Formally, let ρ be a liner order on $\mathcal{N}$; i.e., ρ is a complete, antisymmetric and transitive binary relation on $\mathcal{N}$. We say that the order ρ is *monotonic* if:

(i) for all $S \in \mathcal{N}$ and $i \notin S$, $(S \cup \{i\})\rho S$, and
(ii) for all $S, T \in \mathcal{N}$ and $i \notin S \cup T$, $S\rho T$ implies $(S \cup \{i\})\rho(T \cup \{i\})$.

Theorem 2 *Let f be a rule. Then, f is efficient, consistent, individually rational from equal division and satisfies independence of irrelevant coalitions if and only if f is an extended uniform rule with the property that there exists a monotonic order ρ on $\mathcal{N}$ satisfying the property that for all $(N, \succeq, t) \in \mathcal{P}$,*

(2.a) $c^f(N, \succeq, t)\rho S$ *for all* $S \in AC(N, \succeq, t) \setminus c^f(N, \succeq, t)$.

Proof See Appendix 2. □

Theorem 3 characterizes, for each order σ on $\mathbb{N}$, the extended uniform rule F^σ as the unique efficient, consistent, and individually rational from equal division rule that satisfies order preservation with respect to σ.

Theorem 3 *Let f be a rule and let σ be an order. Then, f is efficient, consistent, individually rational from equal division and satisfies order priority with respect to σ if and only if $f = F^\sigma$.*

Proof See Appendix 3. □

Since, by Remark 1, order priority with respect to σ implies independence of irrelevant coalitions, it follows that the class of rules characterized in Theorem 3 is a subset of the class of rules characterized in Theorem 2.

Before finishing this section we want to point out that in each of the three characterization theorems the set of axioms are independent. See Appendices A1.3, A2.2, and A3.2 for the examples showing their independence.

6 Discussion and final remarks

First, the (large) class of extended uniform rules identified in Theorem 1 satisfy also other appealing properties.

A rule satisfies the property of *independence of irrelevant agents* if at a given problem an agent either receives the zero share or does not participate then, at the problem where the agent is not present anymore, all other agents receive the same share they had received in the original problem. Formally, (Independence of Irrelevant Agents) A rule f is *independent of irrelevant agents* if for each problem $(N, \succeq_N, t)$ such that either $f_i(N, \succeq_N, t) = 0$ or $f_i(N, \succeq_N, t) = \text{NP}$ for some agent $i \in N$ then, $f_j(N, \succeq_N, t) = f_j(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t)$ for all $j \in N\backslash\{i\}$.

A rule satisfies *non-bossiness* if one agent receives the same share at two problems that are identical except for the preferences of this agent then, the shares of all the other agents also coincide at the two problems. Formally,

(Non- bossy) A rule f is *non-bossy* if for each problem $(N, \succeq, t)$, each agent $i \in N$, and each i's preferences $\succeq_i'$ such that $f_i(N, (\succeq_i, \succeq_{N\backslash\{i\}}), t) = f_i(N, (\succeq_i', \succeq_{N\backslash\{i\}}), t)$ then, $f_j(N, (\succeq_i, \succeq_{N\backslash\{i\}}), t) = f_j(N, (\succeq_i', \succeq_{N\backslash\{i\}}), t)$ for all $j \in N\backslash\{i\}$.

A rule satisfies maximality if the set of agents that receive a positive share constitutes (according to set-wise inclusion) a maximal admissible coalition.

(Maximality) A rule is *maximal* if the following holds. Let S be an admissible coalition for the problem $(N, \succeq, t)$ and assume that $\sum_{j \in S} f_j(N, \succeq, t) = t$ and $0 < l_i$ for all $i \in N\backslash S$. Then, for any $T \supsetneq S$, T is not an admissible coalition for $(N, \succeq, t)$.

By condition (1.a) in Theorem 1, all efficient, consistent and individually rational from equal division rules are maximal. Moreover, Remark 2 below states that non-bossyness and independence of irrelevant agents follow from consistency.

Remark 2 Let f be a consistent rule. Then, f is independent of irrelevant agents and non-bossy.

To show that the statement in Remark 2 holds, assume f is consistent. It follows immediately that f is independent of irrelevant agents. To show that f is non-bossy, consider a problem $(N, \succeq_N, t)$, an agent $i \in N$ and a preference $\succeq_i'$ such that,

$$f_i(N, (\succeq_i, \succeq_{N\backslash\{i\}}), t) = f_i(N, (\succeq_i', \succeq_{N\backslash\{i\}}), t). \tag{1}$$

Since f is consistent, for all $j \in N\backslash\{i\}$,

$$\begin{aligned} &f_j(N, (\succeq_i, \succeq_{N\backslash\{i\}}), t) = f_j(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t - f_i(N, (\succeq_i, \succeq_{N\backslash\{i\}}), t)) \quad \text{and} \\ &f_j(N, (\succeq_i', \succeq_{N\backslash\{i\}}), t) = f_j(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t - f_i(N, (\succeq_i', \succeq_{N\backslash\{i\}}), t)). \end{aligned}$$

By (1), $f_j(N, (\succeq_i, \succeq_{N\backslash\{i\}}), t) = f_j(N, (\succeq_i', \succeq_{N\backslash\{i\}}), t)$. Hence, f is non-bossy.

Second, we discuss now why extended uniform rules do not satisfy other appealing properties.

As we have already discussed, extended uniform rules are not strategy-proof. This requirement is too demanding because feasible rules have to depend strongly on agents' intervals of participation which makes them extremely vulnerable to manipulations.

There are other reasonable extensions of Sönmez (1994)'s individual rationality from equal division. For instance, when N is an admissible coalition, one could start allocating the good by preliminary assigning the vector of upper bounds and then decrease uniformly agents' shares (as long as all lower bounds were satisfied) until the total amount of the good would be distributed. This approach would give rise to another set of similar rules. However, they would be different than those rules identified in this article since the two versions of the axiom are in general incompatible.

To see that, consider the problem where $N = \{1, 2\}$, $t = 10$ and R is any profile with $(l_1, p_1, u_1) = (2, 6, 6)$ and $(l_2, p_2, u_2) = (2, 10, 10)$. If a rule satisfies the two versions of the axiom then agent 1 has to receive a share in the interval $[5, 6]$ and agent 2 a share in $[7, 10]$, which is unfeasible.[13]

Thomson (1994a) characterizes the uniform rule in the division problem as the unique single-valued selection satisfying individual rationality from equal division, efficiency, *bilateral consistency* and M-continuity (a requirement needed to select well-behaved rules from correspondences). However, it is not possible to replace consistency in our Theorem 1 by bilateral consistency. The reason is that the choice of the admissible coalition can be made according to bilateral consistency but it may fail to satisfy consistency. For instance, consider the two rules F^{σ^1} and F^{σ^2} where $\sigma^1(i) = i$ for all $i \in \mathbb{N}$ and $\sigma^2(1) = 2$, $\sigma^2(2) = 1$ and $\sigma^2(j) = j$ for all $j > 2$ and define f as follows. For all $(N, \succeq, t) \in \mathcal{P}$,

$$f(N, \succeq, t) = \begin{cases} F^{\sigma^1}(N, \succeq, t) & \text{if } \#N \text{ is odd} \\ F^{\sigma^2}(N, \succeq, t) & \text{if } \#N \text{ is even.} \end{cases}$$

It is easy to see that f satisfies bilateral consistency but it is not consistent.

The non-envy comparison cannot be made when agents' sets of acceptable shares are different. A natural *conditional non-envy* property would require that if agent i's share belongs to agent j's interval of acceptable shares, then agent j should not want to switch. Nevertheless, extended uniform rules do not satisfy conditional no-envy. To see that, consider the problem $(N, \succeq, t)$ where $N = \{1, 2\}, t = 10$, and R is any profile with $(l_1, p_1, u_1) = (2, 10, 10)$ and $(l_2, p_2, u_2) = (0, 10, 10)$. Any extended uniform rule selects at this problem the vector $(6, 4)$ where agent 2 conditionally envies agent 1. The different lower bounds generates asymmetric shares that make conditional envy possible.

Third, the example used to prove that there is no efficient and anonymous rule [(1.2) in the proof of Proposition 1] suggests that random rules may be useful to restore the compatibility of efficiency with fairness properties (like ex-ante equal treatment of equals). However, this approach would require to extend agents' preferences on sure shares to preferences on random shares. We leave for further research a systematic analysis of random rules in this setup.

Finally, in some steps in the proofs of the theorems we use profiles $\succeq$ where agents' intervals of acceptable shares depend on a small number $\varepsilon > 0$ and are degenerated since for all $i \in N, l_i = p_i = u_i$. However, we could also choose $\varepsilon > 0$ in such a way that for each $i \in N$, $\succeq_i$ could be characterized by (l_i, p_i, u_i) with $0 < l_i < p_i < u_i$. However, the case $l_i = p_i = u_i$ makes the arguments more transparent.

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[13] See Chun and Thomson (1990), Schummer and Thomson (1997) and Thomson (1994a,b, 1996) for extensive discussions of the individual rationality requirement in the division problem.

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Appendix 1: Proof of Theorem 1

A1.1. Preliminaries

We first introduce the property of bilateral peaks-and-bounds onlyness. It says that for problems with only two agents at which the set of the two agents is an admissible coalition, the rule depends only on the peaks and the bounds of the two agents' preferences.

(Bilateral peaks- and- bounds only) A rule f is *bilateral peaks-and-bounds only* if for any pair of problems $(N, \succeq, t)$ and $(N, \succeq', t)$ with $|N| = 2$, $N \in \text{AC}(N, \succeq, t)$, and $(l_i, p_i, u_i) = (l'_i, p'_i, u'_i)$ for each $i \in N$, then $f(N, \succeq, t) = f(N, \succeq', t)$.

Before proving Theorem 1 we state and prove three lemmata. The proofs of lemmata 1 and 2 adapt to our setting the corresponding proofs of Lemmata 5 and 6 in Dagan (1996).

Lemma 1 *Let f be an efficient and consistent rule that satisfies individual rationality from equal division. Then, f satisfies bilateral peaks-and-bounds onlyness.*

Proof Let $(N, \succeq, t), (N^*, \succeq^*, t) \in \mathcal{P}$ be such that $N = \{i, j\}$, $N^* = \{k, m\}$, $\{i, j\} \cap \{k, m\} = \varnothing$, $\succeq_i = \succeq^*_k$, $\succeq_j = \succeq^*_m$, $N \in \text{AC}(N, \succeq, t)$, and $N^* \in \text{AC}(N^*, \succeq^*, t)$. Define $x = f(N \cup N^*, (\succeq, \succeq^*), 2t)$. Since N and N^* are admissible at their respective problems, $N \cup N^* \in \text{AC}(N \cup N^*, (\succeq, \succeq^*), 2t)$.

In the rest of the proof of this lemma we make an abuse of notation and we take $x_\alpha = 0$ when $x_\alpha = \text{NP}$ and x_α appears in a sum. Thus, $x_i + x_j + x_k + x_m = 2t$. Since f is consistent,

$$f_i(\{i, k\}, (\succeq_i, \succeq^*_k), 2t - (x_j + x_m)) = x_i \text{ and } f_k(\{i, k\}, (\succeq_i, \succeq^*_k), 2t - (x_j + x_m)) = x_k.$$

Since f satisfies individual rationality from equal division,

$$f_i(\{i, k\}, (\succeq_i, \succeq^*_k), 2t - (x_j + x_m)) = f_k(\{i, k\}, (\succeq_i, \succeq^*_k), 2t - (x_j + x_m)).$$

Thus, $x_i = x_k$. Similarly, we conclude that $x_j = x_m$. Thus, $x_i + x_j = x_k + x_m = t$. By consistency,

$$\begin{aligned} f_i(N, \succeq, t) &= x_i = x_k = f_k(N^*, \succeq^*, t) \text{ and} \\ f_j(N, \succeq, t) &= x_j = x_m = f_m(N^*, \succeq^*, t). \end{aligned} \tag{2}$$

Now, let $\succeq' = (\succeq'_i, \succeq'_j)$ be such that $(l'_i, p'_i, u'_i) = (l_i, p_i, u_i)$ and $(l'_j, p'_j, u'_j) = (l_j, p_j, u_j)$. We want to show that $f(N, \succeq, t) = f(N, \succeq', t)$. Define $x' = f(N \cup N^*, (\succeq', \succeq^*), 2t)$. Using arguments similar to those used above we can conclude that,

$$f_i(N, \succeq', t) = x'_i = x'_k = f_k(N^*, \succeq^*, t) \quad \text{and}$$
$$f_j(N, \succeq', t) = x'_j = x'_m = f_m(N^*, \succeq^*, t).$$

Thus, $f(N, \succeq, t) = f(N, \succeq', t)$. □

Lemma 2 *Let f be an efficient and consistent rule that satisfies individual rationality from equal division and let $(\{i, j\}, \succeq, t) \in \mathcal{P}$ be such that $\{i, j\}$ is an admissible coalition. Then, $f_k(\{i, j\}, \succeq, t) = l_k + F_k(\{i, j\}, \succeq^l, t - l_i - l_j)$ for all $k \in \{i, j\}$.*

Proof Let $(\{i, j\}, \succeq, t) \in \mathcal{P}$ be such that $\{i, j\} \in \mathrm{AC}(\{i, j\}, \succeq, t)$. For each $k \in \{i, j\}$, define $x_k = l_k + \min\{\alpha, u_k - l_k\}$, where $\alpha \in \mathbb{R}$ is such that $x_i + x_j = t$ (as in the definition of individual rationality from equal division applied to the problem $(\{i, j\}, \succeq, t)$). We distinguish between the two rationing situations.

Consider the case $p_i + p_j \leq t$. Assume first that $x_k \geq p_k$ for all $k \in \{i, j\}$. Since f is efficient and satisfies individual rationality from equal division,

$$f_k(\{i, j\}, \succeq, t) = x_k = l_k + F_k(\{i, j\}, \succeq^l, t - l_i - l_j)$$

for all $k \in \{i, j\}$. Without loss of generality assume now that $x_i < p_i$. Thus, $t - x_i = x_j > p_j$. By efficiency, $f_i(\{i, j\}, \succeq, t) \geq p_i > x_i$. Suppose that $f_i(N, \succeq, t) > p_i$. We can find $\succeq'_i$ such that $(l'_i, p'_i, u'_i) = (l_i, p_i, u_i)$ and $x_i \succ'_i f_i(N, \succeq, t)$. By Lemma 1, $f_i(\{i, j\}, \succeq, t) = f_i(\{i, j\}, (\succeq'_i, \succeq_j), t)$. Let $x'_i = l'_i + \min\{\alpha', u'_i - l'_i\}$ be as in the definition of individual rationality from equal division as applied to the problem $(\{i, j\}, (\succeq'_i, \succeq_j), t)$. It is obvious that $x'_i = x_i$. Hence, $x'_i \succ'_i f_i(\{i, j\}, (\succeq'_i, \succeq_j), t)$, which contradicts that f satisfies individual rationality from equal division at the problem $(\{i, j\}, (\succeq'_i, \succeq_j), t)$. Then, $f_i(\{i, j\}, \succeq, t) = p_i = l_i + F_i(\{i, j\}, \succeq^l, t - l_i - l_j)$ and hence, $f_j(\{i, j\}, \succeq, t) = l_j + F_j(\{i, j\}, \succeq^l, t - l_i - l_j)$.

A similar argument can be used to show that the desirable statement also holds when $p_i + p_j > t$. □

Lemma 3 *Let f be an efficient and consistent rule that satisfies individual rationality from equal division. Let $(N, \succeq, t)$ be a problem at which N is an admissible coalition. Then, for each $i \in N$, $f_i(N, \succeq, t) = l_i + F_i(N, \succeq^l, t - \sum_{j \in N} l_j)$.*

Proof Let $(N, \succeq, t)$ be an arbitrary problem with $N \in \mathrm{AC}(N, \succeq, t)$. We proceed by induction on $|N|$. If $|N| = 2$, the result follows from Lemma 2. Assume $|N| > 2$ and suppose that the statement holds for all problems $(N', \succeq', t')$ with $|N'| < |N|$ and $N' \in \mathrm{AC}(N', \succeq', t')$. We prove that it also holds for $(N, \succeq, t)$. For each $i \in N$, define

$$g_i(N, \succeq, t) = l_i + F_i\left(N, \succeq^l, t - \sum_{j \in N} l_j\right).$$

Since N is admissible, by individual rationality from equal division, $\sum_{j \in N} f_j(N, \succeq, t) = t$. To obtain a contradiction, suppose that $f(N, \succeq, t) \neq g(N, \succeq, t)$. Then, there exist $i, j \in N$ such that

$$f_i(N, \succeq, t) > g_i(N, \succeq, t) \quad \text{and} \quad f_j(N, \succeq, t) < g_j(N, \succeq, t). \tag{3}$$

Without loss of generality, assume that $i = 1$ and $j = 2$. Since f is consistent,

$$\begin{aligned} f_i(N, \succeq, t) &= f_i(N\backslash\{1\}, \succeq_{N\backslash\{1\}}, t - f_1(N, \succeq, t)) \quad \text{for all } i \in N\backslash\{1\}, \quad \text{and} \\ f_k(N, \succeq, t) &= f_k(N\backslash\{2\}, \succeq_{N\backslash\{2\}}, t - f_2(N, \succeq, t)) \quad \text{for all } k \in N\backslash\{2\}. \end{aligned} \tag{4}$$

In Lemma 5 in the proof of Theorem 1 below we will show (without using this result) that any extended uniform rule is consistent. Thus,

$$\begin{aligned} g_i(N, \succeq, t) &= g_i(N\backslash\{1\}, \succeq_{N\backslash\{1\}}, t - g_1(N, \succeq, t)) \quad \text{for all } i \in N\backslash\{1\}, \quad \text{and} \\ g_k(N, \succeq, t) &= g_k(N\backslash\{2\}, \succeq_{N\backslash\{2\}}, t - g_2(N, \succeq, t)) \quad \text{for all } k \in N\backslash\{2\}. \end{aligned} \tag{5}$$

By the induction hypothesis, for all $i \in N\backslash\{1\}$,

$$f_i(N\backslash\{1\}, \succeq_{N\backslash\{1\}}, t - f_1(N, \succeq, t)) = g_i(N\backslash\{1\}, \succeq_{N\backslash\{1\}}, t - f_1(N, \succeq, t)). \tag{6}$$

Since $t - f_1(N, \succeq, t) < t - g_1(N, \succeq, t)$, the definition of g implies that for all $i \in N\backslash\{1\}$,

$$g_i(N\backslash\{1\}, \succeq_{N\backslash\{1\}}, t - f_1(N, \succeq, t)) \leq g_i(N\backslash\{1\}, \succeq_{N\backslash\{1\}}, t - g_1(N, \succeq, t)). \tag{7}$$

Hence, by (4)–(7), $f_i(N, \succeq, t) \leq g_i(N, \succeq, t)$ for all $i \in N\backslash\{1\}$. Analogously, $f_k(N, \succeq, t) \geq g_k(N, \succeq, t)$ for all $k \in N\backslash\{2\}$. Thus, $f_i(N, \succeq, t) = g_i(N, \succeq, t)$ for all $i \in N\backslash\{1, 2\}$. Since f and g are consistent, for each $i \in \{1, 2\}$,

$$\begin{aligned} f_i(N, \succeq, t) &= f_i(N\backslash\{1, 2\}, \succeq_{N\backslash\{1,2\}}, t - \textstyle\sum_{j \in \{1,2\}} f_j(N, \succeq, t)), \quad \text{and} \\ g_i(N, \succeq, t) &= g_i(N\backslash\{1, 2\}, \succeq_{N\backslash\{1,2\}}, t - \textstyle\sum_{j \in \{1,2\}} g_j(N, \succeq, t)). \end{aligned}$$

By the induction hypothesis, $f_i(N, \succeq, t) = g_i(N, \succeq, t)$ for all $i \in \{1, 2\}$, a contradiction with (3). □

A1.2. Proof of the characterization

($\Longrightarrow$) Let f be an efficient and consistent rule that satisfies individual rationality from equal division. We first show that f is an extended uniform rule. Let $(N, \succeq, t)$ be an arbitrary problem. By consistency, for each $i \in c^f(N, \succeq, t)$,

$$f_i(N, \succeq, t) = f_i\left(c^f(N, \succeq, t), \succeq_{c^f(N,\succeq,t)}, t\right). \tag{8}$$

Since $c^f(N, \succeq, t)$ is admissible at $(c^f(N, \succeq, t), \succeq_{c^f(N,\succeq,t)}, t)$ and f is efficient, consistent and satisfies individual rationality from equal division we deduce, from Lemma 3, that for all $i \in c^f(N, \succeq, t)$,

$$f_i\left(c^f(N,\succeq,t),\succeq_{c^f(N,\succeq,t)},t\right)$$
$$=l_i+F_i\left(c^f(N,\succeq,t),\succeq^l_{c^f(N,\succeq,t)},t-\sum_{j\in c^f(N,\succeq,t)}l_j\right).$$

Hence, by (8), $f_i(N,\succeq,t)=l_i+F_i(c^f(N,\succeq,t),\succeq^l_{c^f(N,\succeq,t)},t-\sum_{j\in c^f(N,\succeq,t)}l_j)$. Moreover, for each $i\notin c^f(N,\succeq,t)$, $f_i(N,\succeq,t)=\text{NP}$. Thus, f is an extended uniform rule.

To prove that (1.a) holds, let $(N,\succeq,t)$ be a problem at which N is an admissible coalition and take any $i\in N$. By individual rationality from equal division, $f_i(N,\succeq,t)\succeq_i l_i+\min\{\alpha,u_i-l_i\}\in[l_i,u_i]$. By definition of $c^f(N,\succeq,t)$, $i\in c^f(N,\succeq,t)$. Since $i\in N$ was arbitrary, $c^f(N,\succeq,t)=N$. Thus, (1.a) holds.

To prove that (1.b) holds, let $(N,\succeq,t)$ be a problem and consider any nonempty $S\subset N$. Since f is consistent, $f_j(N,\succeq,t)=f_j(S,\succeq_S,t-\sum_{i\in c^f(N,\succeq,t)\setminus S}f_i(N,\succeq,t))$ for each $j\in S$. Now,

$$c^f(S,\succeq_S,t-\textstyle\sum_{i\in c^f(N,\succeq,t)\setminus S}f_i(N,\succeq,t))$$
$$=\left\{j\in S\mid f_j\left(S,\succeq_S,t-\textstyle\sum_{i\in c^f(N,\succeq,t)\setminus S}f_i(N,\succeq,t)\right)\in[l_j,u_j]\right\}$$
$$=\{j\in S\mid f_j(N,\succeq,t)\in[l_j,u_j]\}$$
$$=c^f(N,\succeq,t)\setminus\cap S.$$

Thus, (1.b) holds.

($\Longleftarrow$) Assume that f is an extended uniform rule that satisfies (1.a) and (1.b). We want to show that f is efficient, consistent and satisfies individual rationality from equal division. We do it by proving Lemmata 4–8 below.

Lemma 4 *The rule F is efficient and consistent on the subdomain of problems $(N,\succeq,t)$ where $l_i=0$ for all $i\in N$ and $N\in AC(N,\succeq,t)$.*

Proof We first prove that $F(N,\succeq,t)$ is Pareto optimal by distinguishing between the two rationing situations.

Assume first that $\sum_{j\in N}p_j<t$. Then, $F_i(N,\succeq,t)=\min\{\max\{\beta,p_i\},u_i\}$ for all $i\in N$. Let $x=(x_i)_{i\in N}\in\text{FA}(N,\succeq,t)$ be such that $x_i\succeq_i F_i(N,\succeq,t)$ for all $i\in N$. It is obvious that $\sum_{j\in N}x_j=t$. We prove that $x_i=F_i(N,\succeq,t)$ for all $i\in N$ by distinguishing among three possible cases.

Case 1 $F_i(N,\succeq,t)=p_i$. Since $x_i\succeq_i F_i(N,\succeq,t)$, $x_i=p_i$.

Case 2 $F_i(N,\succeq,t)=u_i$. Since $x_i\succeq_i F_i(N,\succeq,t)$, $x_i\le u_i$. Suppose that $x_i<u_i$. As $\sum_{j\in N}x_j=\sum_{j\in N}F_j(N,\succeq,t)=t$, there exists $k\in N$ such that $x_k>F_k(N,\succeq,t)$. By its definition, $F_k(N,\succeq,t)$ can only take three different values. If $F_k(N,\succeq,t)=u_k$ then, $x_k>u_k$ which contradicts that $x\in\text{FA}(N,\succeq,t)$. If $F_k(N,\succeq,t)=p_k$ then, $x_k>p_k$ which contradicts that $x_k\succeq_k F_k(N,\succeq,t)$. Finally, if $F_k(N,\succeq,t)=\beta$ and $p_k<\beta<u_k$ then, $\beta<x_k$. Since $x\in\text{FA}(N,\succeq,t)$, $x_k\le u_k$, which contradicts, by (P.2), that $x_k\succeq_k F_k(N,\succeq,t)$. Thus, $x_i=u_i$.

Case 3 $F_i(N, \succeq, t) = \beta$ and $\beta > p_i$ (if $\beta = p_i$, apply Case 1 above). Since $x_i \succeq_i F_i(N, \succeq, t)$, $x_i \leq \beta$ by (P.2). Suppose that $x_i < \beta$. As $\sum_{j\in N} x_j = \sum_{j\in N} F_j(N, \succeq, t) = t$, there exists $k \in N$ such that $x_k > F_k(N, \succeq, t)$. Using arguments similar to those already used in Case 2 we obtain a contradiction. Thus, $x_i = \beta$.

A similar argument can be used to show that $F(N, \succeq, t)$ is Pareto optimal when $\sum_{j\in N} p_j \geq t$ (and $F_i(N, \succeq, t) = \min\{\beta, p_i\}$ for all $i \in N$).

To prove that F is consistent, it is sufficient to show that for all $i \in N\backslash\{k\}$, $F_i(N, \succeq, t) = F_i(N\backslash\{k\}, \succeq_{N\backslash\{k\}}, t - f_k(N, \succeq, t))$ for any arbitrary agent $k \in N$. Again, we distinguish between the two rationing situations.

Assume first that $\sum_{j\in N} p_j < t$. Then, $F_i(N, \succeq, t) = \min\{\max\{\beta, p_i\}, u_i\}$ for all $i \in N$. Thus, $p_i \leq F_i(N, \succeq, t)$ for all $i \in N$. Let $k \in N$. Then, $\sum_{j\in N\backslash\{k\}} p_j \leq \sum_{j\in N\backslash\{k\}} F_j(N, \succeq, t)$. We distinguish between two possible cases.

Case 1 $\sum_{j\in N\backslash\{k\}} p_j < \sum_{j\in N\backslash\{k\}} F_j(N, \succeq, t) = t - F_k(N, \succeq, t)$. Since

$$\sum_{j\in N\backslash\{k\}} \min\{\max\{\beta, p_j\}, u_j\} = t - F_k(N, \succeq, t),$$

and $F_i(N\backslash\{k\}, \succeq_{N\backslash\{k\}}, t - F_k(N, \succeq, t)) = \min\{\max\{\beta', p_i\}, u_i\}$ where β' is the unique number satisfying

$$\sum_{j\in N\backslash\{k\}} \min\{\max\{\beta', p_j\}, u_j\} = t - F_k(N, \succeq, t),$$

we deduce that $\beta = \beta'$ and, for each $i \in N\backslash\{k\}$

$$F_i(N\backslash\{k\}, \succeq_{N\backslash\{k\}}, t - F_k(N, \succeq, t)) = \min\{\max\{\beta, p_i\}, u_i\} = F_i(N, \succeq, t).$$

Case 2 $\sum_{j\in N\backslash\{k\}} p_j = \sum_{j\in N\backslash\{k\}} F_j(N, \succeq, t) = t - F_k(N, \succeq, t)$. Then, by efficiency of F, $F_i(N, \succeq, t) = p_i$ for all $i \in N\backslash\{k\}$. Moreover, for each $i \in N\backslash\{k\}$,

$$F_i(N\backslash\{k\}, \succeq_{N\backslash\{k\}}, t - F_k(N, \succeq, t)) = \min\{\beta, p_i\},$$

where β is the unique number satisfying

$$\sum_{j\in N\backslash\{k\}} \min\{\beta, p_j\} = t - F_k(N, \succeq, t) = \sum_{j\in N\backslash\{k\}} p_j.$$

Thus, $\beta = \max_{j\in N\backslash\{k\}}\{p_j\}$. Hence, for each $i \in N\backslash\{k\}$,

$$F_i(N\backslash\{k\}, \succeq_{N\backslash\{k\}}, t - F_k(N, \succeq, t)) = p_i.$$

The case $\sum_{j\in N} p_j \geqslant t$ is similar and we omit it.

□

Lemma 5 *The rule f is consistent.*

Proof Let $(N, \succeq, t) \in \mathcal{P}$ and $S \subsetneq N$. We have to show that for all $i \in S$,

$$f_i(N, \succeq, t) = f_i(S, \succeq_S, t - \textstyle\sum_{j \in c^f(N,\succeq,t)\setminus S} f_j(N, \succeq, t)).$$

It is sufficient to prove that it holds for $|S| = n - 1$. Let $k \in N$ and $i \in N\setminus\{k\}$. We distinguish between two cases.

Case 1 $i \notin c^f(N, \succeq, t)$. Then, $f_i(N, \succeq, t) = \text{NP}$. By (1.b), $c^f(N\setminus\{k\}, \succeq_{N\setminus\{k\}}, \bar{t}) = c^f(N, \succeq, t)\setminus\{k\}$, where

$$\bar{t} = \begin{cases} t & \text{if } k \notin c^f(N, \succeq, t) \\ t - f_k(N, \succeq, t) & \text{otherwise.} \end{cases}$$

Hence, $i \notin c^f(N\setminus\{k\}, \succeq_{N\setminus\{k\}}, \bar{t})$ and then,

$$f_i(N\setminus\{k\}, \succeq_{N\setminus\{k\}}, \bar{t}) = \text{NP} = f_i(N, \succeq, t).$$

Case 2 $i \in c^f(N, \succeq, t)$. Since by hypothesis f is an extended uniform rule,

$$f_i(N, \succeq, t) = l_i + F_i(c^f(N, \succeq, t), \succeq^l_{c^f(N,\succeq,t)}, t - \textstyle\sum_{j \in c^f(N,\succeq,t)} l_j).$$

By (1.b), $i \in c^f(N\setminus\{k\}, \succeq_{N\setminus\{k\}}, \bar{t}) = c^f(N, \succeq, t)\setminus\{k\}$. Then,

$$\begin{aligned} &f_i(N\setminus\{k\}, \succeq_{N\setminus\{k\}}, \bar{t}) \\ &\quad = l_i + F_i\left(c^f(N, \succeq, t)\setminus\{k\}, \succeq^l_{c^f(N,\succeq,t)\setminus\{k\}}, \bar{t} - \textstyle\sum_{j \in c^f(N,\succeq,t)\setminus\{k\}} l_j\right). \end{aligned}$$

We consider two subcases.

Subcase 2.1 $k \notin c^f(N, \succeq, t)$. Then, $f_k(N, \succeq, t) = \text{NP}$ and $\bar{t} = t$. Now,

$$\begin{aligned} &F_i\left(c^f(N, \succeq, t)\setminus\{k\}, \succeq^l_{c^f(N,\succeq,t)\setminus\{k\}}, \bar{t} - \textstyle\sum_{j \in c^f(N,\succeq,t)\setminus\{k\}} l_j\right) \\ &\quad = F_i\left(c^f(N, \succeq, t), \succeq^l_{c^f(N,\succeq,t)}, t - \textstyle\sum_{j \in c^f(N,\succeq,t)} l_j\right). \end{aligned}$$

Hence,

$$f_i(N\setminus\{k\}, \succeq_{N\setminus\{k\}}, t) = f_i(N, \succeq, t).$$

Subcase 2.2 $k \in c^f(N, \succeq, t)$. By Lemma 4, F is consistent on the smaller subdomain. Thus, setting $c^f \equiv c^f(N, \succeq, t)$,

$$\begin{aligned} F_i\left(c^f, \succeq^l_{c^f}, t - \textstyle\sum_{j \in c^f} l_j\right) = F_i\Big(&c^f\setminus\{k\}, \succeq^l_{c^f\setminus\{k\}}, t - \textstyle\sum_{j \in c^f} l_j \\ &- F_k(c^f, \succeq^l_{c^f}, t - \textstyle\sum_{j \in c^f} l_j)\Big). \end{aligned} \tag{9}$$

Since $k \in c^f$ and f is an extended uniform rule,

$$-l_k - F_k\left(c^f, \succeq^l_{c^f}, t - \sum_{j \in c^f} l_j\right) = -f_k(N, \succeq, t). \tag{10}$$

Now, by (9) and (10),

$$\begin{aligned} f_i(N, \succeq, t) &= l_i + F_i\left(c^f, \succeq^l_{c^f}, t - \sum_{j \in c^f} l_j\right) \\ &= F_i\left(c^f \backslash \{k\}, \succeq^l_{c^f \backslash \{k\}}, t - \sum_{j \in c^f} l_j - F_k(c^f, \succeq^l_{c^f}, t - \sum_{j \in c^f} l_j)\right) \\ &= l_i + F_i\left(c^f \backslash \{k\}, \succeq^l_{c^f \backslash \{k\}}, t - \sum_{j \in c^f \backslash \{k\}} l_j - f_k(N, \succeq, t)\right) \\ &= f_i\left(N \backslash \{k\}, \succeq_{N \backslash \{k\}}, t - f_k(N, \succeq, t)\right), \end{aligned}$$

where the last equality follows from the definition of extended uniform rules. □

Lemma 6 *The rule f satisfies individual rationality from equal division.*

Proof Let $(N, \succeq, t)$ be such that N is an admissible coalition. By (1.a), $c^f(N, \succeq, t) = N$. Since f is an extended uniform rule,

$$f_i(N, \succeq, t) = l_i + F_i\left(N, \succeq^l_N, t - \sum_{j \in N} l_j\right)$$

for all $i \in N$. We will show that for all $i \in N$,

$$f_i(N, \succeq, t) \succeq_i l_i + \min\{\alpha, u_i - l_i\},$$

where $\sum_{j \in N} \min\{\alpha, u_j - l_j\} = t - \sum_{j \in N} l_j$.

Assume first that $\sum_{j \in N} p_j < t$. Then, $\sum_{j \in N}(p_j - l_j) < t - \sum_{j \in N} l_j$. Now, for each $i \in N$,

$$F_i(N, \succeq^l, t - \sum_{j \in N} l_j) = \min\{\max\{\beta, p_i - l_i\}, u_i - l_i\},$$

where β is the unique number satisfying $\sum_{j \in N} \min\{\max\{\beta, p_j - l_j\}, u_j - l_j\} = t - \sum_{j \in N} l_j$. Then, $\alpha \geq \beta$ because

$$\sum_{j \in N} \min\{\max\{\alpha, p_j - l_j\}, u_j - l_j\} \geq \sum_{j \in N} \min\{\alpha, u_j - l_j\} = t - \sum_{j \in N} l_j.$$

Let $i \in N$. We consider separately the following three cases:

Case 1 $\min\{\max\{\beta, p_i - l_i\}, u_i - l_i\} = p_i - l_i$. Then, $f_i(N, \succeq, t) = p_i$ and $f_i(N, \succeq, t) \succsim_i l_i + \min\{\alpha, u_i - l_i\}$.

Case 2 $\min\{\max\{\beta, p_i - l_i\}, u_i - l_i\} = u_i - l_i > p_i - l_i$. Then,

$$\begin{aligned} \min\{\alpha, u_i - l_i\} &= u_i - l_i \\ f_i(N, \succeq, t) &= l_i + (u_i - l_i) = u_i, \quad \text{and} \\ l_i + \min\{\alpha, u_i - l_i\} &= u_i. \end{aligned}$$

Thus, $f_i(N, \succeq, t) \sim_i l_i + \min\{\alpha, u_i - l_i\}$.

Case 3 $\min\{\max\{\beta, p_i - l_i\}, u_i - l_i\} = \beta > p_i - l_i$. We consider two subcases.

Subcase 3.1 $\alpha \leq u_i - l_i$. Then,

$$\begin{aligned} \min\{\alpha, u_i - l_i\} &= \alpha, \\ f_i(N, \succeq, t) &= l_i + \beta, \quad \text{and} \\ l_i + \min\{\alpha, u_i - l_i\} &= l_i + \alpha. \end{aligned}$$

Since $l_i + \alpha \geq l_i + \beta \geq p_i$, by (P.2), $f_i(N, \succeq, t) \succeq_i l_i + \min\{\alpha, u_i - l_i\}$.
Subcase 3.2 $\alpha > u_i - l_i$. Then,

$$\begin{aligned} \min\{\alpha, u_i - l_i\} &= u_i - l_i, \\ f_i(N, \succeq, t) &= l_i + \beta, \quad \text{and} \\ l_i + \min\{\alpha, u_i - l_i\} &= l_i + u_i - l_i = u_i. \end{aligned}$$

Since $p_i < l_i + \beta = f_i(N, \succeq, t) \leq u_i$, by (P.2), $f_i(N, \succeq, t) \succeq_i l_i + \min\{\alpha, u_i - l_i\}$.

The case $\sum_{j \in N} p_j \geq t$ is similar and we omit it. □

Lemma 7 *The rule f is efficient.*

Proof Suppose not. Then, there exist $(N, \succeq, t) \in \mathcal{P}$, $x \in \text{FA}(N, \succeq, t)$, and $j \in N$ such that $x_i \succeq_i f_i(N, \succeq, t)$ for all $i \in N$ and $x_j \succ_j f_j(N, \succeq, t)$. Since $x_j \succ_j f_j(N, \succeq, t)$, $j \in S(x)$ (the set of agents $k \in N$ such that $l_k \leq x_k \leq u_k$) and hence $S(x) \neq \varnothing$. Moreover, $c^f(N, \succeq, t) \subset S(x)$. Since $S(x) \neq \varnothing$ is an admissible coalition at $(N, \succeq, t)$, $c^f(N, \succeq, t) \neq \varnothing$.

Since f satisfies consistency and $c^f(N, \succeq, t) \subset S(x)$, $f_i(S(x), \succeq_{S(x)}, t) = f_i(N, \succeq, t)$ for all $i \in S(x)$. By (1.*a*), $c^f(S(x), \succeq_{S(x)}, t) = S(x)$. By (1.*b*), $c^f(S(x), \succeq_{S(x)}, t) = c^f(N, \succeq, t) \cap S(x)$. Thus, $S(x) = c^f(N, \succeq, t)$. Now $(x_i - l_i)_{i \in c^f(N, \succeq, t)}$ Pareto dominates

$$\left(F_i(c^f(N, \succeq, t), (\succeq_j^l)_{j \in c^f(N, \succeq, t)}, t - \textstyle\sum_{j \in c^f(N, \succeq, t)} l_j)\right)_{i \in c^f(N, \succeq, t)},$$

which contradicts Lemma 4. □

This finishes the proof of the characterization in Theorem 1.

A1.3. Independence of the axioms

Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be the identity order; i.e., $\sigma(i) = i$ for all $i \in \mathbb{N}$.

Consider the rule f^1 defined as follows. Given $(N, \succeq, t) \in \mathcal{P}$, set $c^{f^1}(N, \succeq, t) = c^{F^\sigma}(N, \succeq, t)$ and

$$f_i^1(N, \succeq, t) = \begin{cases} \text{NP} & \text{if } i \notin c^{f^1}(N, \succeq, t) \\ l_i + \min\{\alpha, u_i - l_i\} & \text{if } i \in c^{f^1}(N, \succeq, t), \end{cases}$$

where $\alpha \in \mathbb{R}$ satisfies $\sum_{j \in c^{f^1}(N,\succeq,t)} \min\{\alpha, u_j - l_j\} = t - \sum_{j \in c^{f^1}(N,\succeq,t)} l_j$. It is not difficult to prove that f^1 is consistent, satisfies individual rationality from equal division, but it is not efficient.

Consider the rule f^2 defined as follows. Given $(N, \succeq, t) \in \mathcal{P}$, set $c^{f^2}(N, \succeq, t) = c^{F^\sigma}(N, \succeq, t)$ and

$$f_i^2(N,\succeq,t) = \begin{cases} \text{NP} & \text{if } i \notin c^{f^2}(N,\succeq,t) \\ D_i^\sigma(c^{f^2}(N,\succeq,t), \succeq_{c^{f^2}(N,\succeq,t)}, t) & \text{if } i \in c^{f^2}(N,\succeq,t), \end{cases}$$

where $D_i^\sigma(c^{f^2}(N,\succeq,t), \succeq_{c^{f^2}(N,\succeq,t)}, t)$ denotes the sequential dictatorial rule induced by the order σ in the problem $(c^{f^2}(N,\succeq,t), \succeq_{c^{f^2}(N,\succeq,t)}, t)$. In the sequential dictatorial rule agents select, following the order σ, the shares they most prefer, as long as there is enough amount of the good (we skip its formal definition). It is not difficult to prove that f^2 is efficient, consistent but it is not individually rational from equal division.

Let $\sigma' : \mathbb{N} \longrightarrow \mathbb{N}$ be any order different from σ. Consider the rule f^3 defined as follows. First, define $f^{1,\sigma'}$ similarly to f^1 but using order σ' instead of σ. Now, for all $(N, \succeq, t) \in \mathcal{P}$,

$$f^3(N,\succeq,t) = \begin{cases} f^1(N,\succeq,t) & \text{if } |N| \text{ is odd} \\ f^{1,\sigma'}(N,\succeq,t) & \text{if } |N| \text{ is even.} \end{cases}$$

It is not difficult to prove that f^3 is efficient, satisfies individual rationality from equal division but it is not consistent.

Appendix 2: Proof of Theorem 2

A2.1. Proof of the characterization

($\Longleftarrow$) We first prove that if f is an extended uniform rule with the property that there exists a monotonic order ρ on $\mathcal{N}$ such that (2.a) holds then, f is efficient, consistent, individually rational from equal division and satisfies independence of irrelevant coalitions. We do it by proving Lemmata 8 and 9 below.

Lemma 8 *The rule f is efficient, consistent and satisfies individual rationality from equal division.*

Proof By Theorem 1, it is sufficient to prove that f satisfies (1.a) and (1.b). We first show that f satisfies (1.a). Let $(N, \succeq, t) \in \mathcal{P}$ be such that N is admissible and let ρ be the monotonic order on $\mathcal{N}$ associated to f. By property (i) of ρ, $N \rho S$ for all $S \subsetneq N$. Thus, $c^f(N, \succeq, t) = N$.

Let $i \in N$. Using an iterated argument it is sufficient to show that f satisfies (1.b) for $S = N \backslash \{i\}$. Let $(N, \succeq, t) \in \mathcal{P}$. We consider separately the following two cases.

Case 1 $i \notin c^f(N, \succeq, t)$. Then, $f_i(N, \succeq, t) = \text{NP}$. Obviously,

$$c^f(N, \succeq, t) \in \text{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t)$$

and

$$\text{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t) \subseteq \text{AC}(N, \succeq, t).$$

By (2.a), $c^f(N, \succeq, t)\rho S$ for all $S \in \text{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t)\backslash c^f(N, \succeq, t)$, which means that

$$\begin{aligned} c^f(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t) &= c^f(N, \succeq, t) \\ &= c^f(N, \succeq, t)\backslash\{i\}. \end{aligned}$$

Case 2 $i \in c^f(N, \succeq, t)$. Then, $f_i(N, \succeq, t) \in [l_i, u_i]$. It is easy to see that

$$S \in \text{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t - f_i(N, \succeq, t)) \quad \text{implies } S \cup \{i\} \in \text{AC}(N, \succeq, t). \tag{11}$$

Moreover, $c^f(N, \succeq, t)\backslash\{i\} \in \text{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t - f_i(N, \succeq, t))$ holds. We prove that $(c^f(N, \succeq, t)\backslash\{i\})\rho S$ for all $S \in \text{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t - f_i(N, \succeq, t))\backslash(c^f(N, \succeq, t)\backslash\{i\})$. Suppose not; there exists $S' \in \text{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t - f_i(N, \succeq, t))$ such that $S'\rho(c^f(N, \succeq, t)\backslash\{i\})$. By (11), $S' \cup \{i\} \in \text{AC}(N, \succeq, t)$. By property (ii) of ρ, $(S' \cup \{i\})\rho c^f(N, \succeq, t)$, which contradicts (2.a). □

Lemma 9 *The rule f satisfies independence of irrelevant coalitions.*

Proof Let $(N, \succeq, t)$ and $(N', \succeq', t')$ be any two problems with the property that $\text{AC}(N', \succeq', t') \subseteq \text{AC}(N, \succeq, t)$ and $c^f(N, \succeq, t) \in \text{AC}(N', \succeq', t')$. By (2.a), for all $S \in \text{AC}(N, \succeq, t)\backslash c^f(N, \succeq, t)$, $c^f(N, \succeq, t)\rho S$ holds. Since $\text{AC}(N', \succeq', t') \subseteq \text{AC}(N, \succeq, t)$ and $c^f(N, \succeq, t) \in A(N', \succeq', t')$, $c^f(N, \succeq, t)\rho S$ for all $S \in \text{AC}(N', \succeq', t')\backslash c^f(N, \succeq, t)$. By (2.a), $c^f(N', \succeq', t') = c^f(N, \succeq, t)$. □

($\Longrightarrow$) Let f be an efficient and consistent rule that satisfies individual rationality from equal division and independent of irrelevant coalitions. By Theorem 1, f is an extended uniform rule. We want to show that there exists a monotonic order ρ on $\mathcal{N}$ such that f satisfies (2.a).

We first define (using f) a binary relation ρ on $\mathcal{N}$. Let $S, S' \in \mathcal{N}$. Three cases are possible.

Case 1 $S \supset S'$. Then, set $S\rho S'$.

Case 2 $S' \supset S$. Then, set $S'\rho S$.

Case 3 There exist agents $j \in S\backslash S'$ and $j' \in S'\backslash S$. Consider any problem $(N, \succeq, t)$ where $S, S' \subseteq N$ and for each $i \in N$, $l_i = p_i = u_i$, and

$$p_i = \begin{cases} \varepsilon & \text{if } i \in S \cap S' \\ \varepsilon^2 & \text{if } i \in S\backslash(S' \cup \{j\}) \\ t - \varepsilon|S \cap S'| - \varepsilon^2|S\backslash(S' \cup \{j\})| & \text{if } i = j \\ \varepsilon^3 & \text{if } i \in S'\backslash(S \cup \{j'\}) \\ t - \varepsilon|S \cap S'| - \varepsilon^3|S'\backslash(S \cup \{j'\})| & \text{if } i = j' \\ \varepsilon^4 & \text{if } i \in N\backslash(S \cup S'). \end{cases}$$

Moreover, we choose $\varepsilon > 0$ small enough to make sure that $0 < p_i < t$ for all $i \in N$ and $\mathrm{AC}(N, \succeq, t) = \{\varnothing, S, S'\}$. Observe that such $\varepsilon > 0$ exists. Thus, $c^f(N, \succeq, t) \in \{S, S'\}$. Then, if $c^f(N, \succeq, t) = S$ set $S\rho S'$ and if $c^f(N, \succeq, t) = S'$ set $S'\rho S$.

Since f satisfies independence of irrelevant coalitions, ρ does not depend on $(N, \succeq, t)$. Namely, let $(N', \succeq', t')$ be such that $\mathrm{AC}(N', \succeq', t') = \{\varnothing, S, S'\}$. Then, $c^f(N', \succeq', t') = c^f(N, \succeq, t)$. Thus, ρ is well defined.

Lemma 10 *If $S\rho S'$ and $T \subset S \cap S'$ then, $(S\backslash T)\rho(S'\backslash T)$.*

Proof If $S \supset S'$ then, the statement follows immediately. Assume $S\backslash S' \neq \varnothing$ and $S'\backslash S \neq \varnothing$ hold. Let $i \in T \subset S \cap S'$ and $(N, \succeq, t)$ be a problem as in the definition of ρ applied to S and S'. Thus, $\mathrm{AC}(N, \succeq, t) = \{\varnothing, S, S'\}$, $c^f(N, \succeq, t) = S$ and $\mathrm{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, \bar{t}) = \{\varnothing, S\backslash\{i\}, S'\backslash\{i\}\}$, where again,

$$\bar{t} = \begin{cases} t & \text{if } k \notin c^f(N, \succeq, t) \\ t - f_k(N, \succeq, t) & \text{otherwise.} \end{cases}$$

Since f satisfies (1.b),

$$c^f(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, \bar{t}) = c^f(N, \succeq, t)\backslash\{i\} = S\backslash\{i\}.$$

Let $(N, \succeq', t)$ be as in the definition of ρ applied to $S\backslash\{i\}$ and $S'\backslash\{i\}$. Thus,

$$\mathrm{AC}(N, \succeq', t) = \{\varnothing, S\backslash\{i\}, S'\backslash\{i\}\} = \mathrm{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, \bar{t}).$$

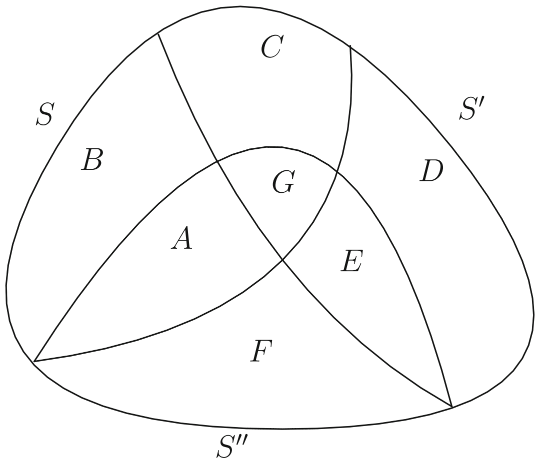
Since f satisfies independence of irrelevant coalitions and $c^f(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, \bar{t}) = S\backslash\{i\}$, $c^f(N, \succeq', t) = S\backslash\{i\}$. Thus, $(S\backslash\{i\})\rho(S'\backslash\{i\})$. Repeating successively the same argument for each agent in $T\backslash\{i\}$ it follows that $(S\backslash T)\rho(S'\backslash T)$. □

Lemma 11 *The binary relation ρ on $\mathcal{N}$ is complete, antisymmetric, and satisfies properties (i) and (ii).*

Proof By definition, ρ is a complete and antisymmetric binary relation. Property (i) holds trivially. Suppose that ρ does not satisfy property (ii). Then, there exist $S, T \subset N$ and $i \in N\backslash(S \cup T)$ such that $S\rho T$ but $(S \cup \{i\})\rho(T \cup \{i\})$ does not hold. Since ρ is complete, $(T \cup \{i\})\rho(S \cup \{i\})$. By Lemma 10, $T\rho S$, which is a contradiction. □

Lemma 12 *The rule f satisfies* (2.a).

Fig. 2



Proof Let $S \in \mathrm{AC}(N, \succeq, t) \backslash c^f(N, \succeq, t)$. We want to prove that $c^f(N, \succeq, t)\rho S$. We distinguish among the following three cases.

Case 1 $S \subsetneq c^f(N, \succeq, t)$. Then $c^f(N, \succeq, t)\rho S$ by definition of ρ.

Case 2 $c^f(N, \succeq, t) \subsetneq S$. We will obtain a contradiction. Consider the problem $(S, \succeq_S, t)$. Since $S \in \mathrm{AC}(N, \succeq, t)$, $S \in \mathrm{AC}(S, \succeq_S, t)$. By Theorem 1, f satisfies (1.a). Thus, $c^f(S, \succeq_S, t) = S$. Since $c^f(N, \succeq, t) \subsetneq S$, $c^f(N, \succeq, t) \in \mathrm{AC}(S, \succeq_S, t)$. Moreover, $\mathrm{AC}(S, \succeq_S, t) \subseteq \mathrm{AC}(N, \succeq, t)$. Since f satisfies independence of irrelevant coalitions, $c^f(S, \succeq_S, t) = c^f(N, \succeq, t)$, a contradiction with $c^f(S, \succeq_S, t) = S$.

Case 3 $c^f(N, \succeq, t) \backslash S \neq \varnothing$ and $S \backslash c^f(N, \succeq, t) \neq \varnothing$. Let $(N, \succeq', t')$ be as in the definition of ρ applied to the sets $c^f(N, \succeq, t)$ and S. Thus, $\mathrm{AC}(N, \succeq', t') = \{c^f(N, \succeq, t), S\}$. Since f satisfies independence of irrelevant coalitions, $c^f(N, \succeq', t') = c^f(N, \succeq, t)$. By the definition of ρ, $c^f(N, \succeq, t)\rho$. □

Lemma 13 below states that ρ is transitive, the only remaining property to be proven in order to finish the proof of the characterization of Theorem 2.

Lemma 13 *The binary relation ρ on $\mathcal{N}$ is transitive.*

Proof To simplify the notation, given a family $\{X_1, X_2, \ldots, X_K\}$ of subsets of N, we denote $\cup_{k=1}^{K} X_k$ by $X_1 X_2 \cdots X_K$. Assume that $S\rho S'$ and $S'\rho S''$. We must prove that $S\rho S''$. We decompose S, S', and S'' according to Fig. 2, with $S = ABCG$, $S' = CDEG$ and $S'' = AEFG$, and prove Claims 1–5 below.

Claim 1 *Assume that $AC(N, \succeq, t) = \{X_k\}_{k=1}^{K}$ and for each $k \neq 1$, there exists j_k such that $X_{j_k}\rho X_k$. Then, $X_1 \rho X_k$ for each $k \neq 1$.*

Proof Since $\mathrm{AC}(N, \succeq, t) \neq \varnothing$, we have that $c^f(N, \succeq, t) \in \mathrm{AC}(N, \succeq, t)$. Let $k \neq 1$ and assume $X_{j_k}\rho X_k$. Since f satisfies (2.a), $c^f(N, \succeq, t) \neq X_k$. Thus, $c^f(N, \succeq, t) = X_1$. Since f satisfies (2.a), $X_1 \rho X_k$ for each $k \neq 1$. □

Claim 2 *Assume that $B \neq \varnothing$, $D \neq \varnothing$, and $F \neq \varnothing$. Then, $S\rho S''$.*

Proof By assumption, for each $X \in \{B, D, F\}$, we can find $i_X \in X$. Consider any problem $(N, \succeq, 1)$ where $BDF \subseteq N$ and for all $i \in N$, $l_i = p_i = u_i$ and

$$p_i = \begin{cases} \varepsilon & \text{if } i \in G \\ \varepsilon^2 & \text{if } i \in C \\ \varepsilon^3 & \text{if } i \in A \\ \varepsilon^4 & \text{if } i \in E \\ \varepsilon^5 & \text{if } i \in B\backslash\{i_B\} \\ 1-\varepsilon|G|-\varepsilon^2|C|-\varepsilon^3|A|-\varepsilon^5|B\backslash\{i_B\}| & \text{if } i = i_B \\ \varepsilon^6 & \text{if } i \in D\backslash\{i_D\} \\ 1-\varepsilon|G|-\varepsilon^2|C|-\varepsilon^4|E|-\varepsilon^6|D\backslash\{i_D\}| & \text{if } i = i_D \\ \varepsilon^7 & \text{if } i \in F\backslash\{i_F\} \\ 1-\varepsilon|G|-\varepsilon^2|C|-\varepsilon^4|E|-\varepsilon^7|F\backslash\{i_F\}| & \text{if } i = i_F \\ 2 & \text{otherwise.} \end{cases}$$

For $\varepsilon > 0$ sufficiently small, $\text{AC}(N, \succeq, 1) = \{\varnothing, S, S', S''\}$. By Claim 1, $c^f(N, \succeq, 1) = S$. Since f satisfies (2.a), $S\rho S''$. □

Claim 3 *Let U, U', V, V' be such that $X \cap Y = \varnothing$ for each $X, Y \in \{U, U', V, V'\}$ with $X \neq Y$ and assume $U\rho U'$ and $V\rho V'$. Then, $UV\rho U'V'$.*

Proof Since $U\rho U'$ and $V\rho V'$, $U \neq \varnothing$ and $V \neq \varnothing$ hold. We consider four cases separately.

Case 1 $U' = V' = \varnothing$. Obviously, $UV\rho\varnothing$.

Case 2 $U' \neq \varnothing$ and $V' \neq \varnothing$. For each $X \in \{U, U', V, V'\}$, take $i_X \in X$. Consider any problem $(N, \succeq, 3)$ where $UU'VV' \subseteq N$, and for all $i \in N$, $l_i = p_i = u_i$ and

$$p_i = \begin{cases} \varepsilon & \text{if } i \in U\backslash\{i_U\} \\ 2-\varepsilon|U\backslash\{i_U\}| & \text{if } i = i_U \\ \varepsilon^2 & \text{if } i \in U'\backslash\{i_{U'}\} \\ 2-\varepsilon^2|U'\backslash\{i_{U'}\}| & \text{if } i = i_{U'} \\ \varepsilon^3 & \text{if } i \in V\backslash\{i_V\} \\ 1-\varepsilon^3|V\backslash\{i_V\}| & \text{if } i = i_V \\ \varepsilon^4 & \text{if } i \in V'\backslash\{i_{V'}\} \\ 1-\varepsilon^4|V'\backslash\{i_{V'}\}| & \text{if } i = i_{V'} \\ 4 & \text{otherwise.} \end{cases}$$

It is easy to see that, for $\varepsilon > 0$ is sufficiently small, $\text{AC}(N, \succeq, 3) = \{\varnothing, UV, UV', U'V, U'V'\}$. Since $U\rho U'$ and, by Lemma 11, ρ satisfies property (ii), $UV\rho U'V$ and $UV'\rho U'V'$. Since $V\rho V'$, and again by property (ii), $UV\rho UV'$. Claim 1 implies $UV\rho U'V'$.

Case 3 $U' \neq \varnothing$ and $V' = \varnothing$. For each $X \in \{U, U', V\}$, take $i_X \in X$. Consider any problem $(N, \succeq, 1)$ where $UU'V \subseteq N$ for all $i \in N\backslash\{i_U\}$, $l_i = p_i = u_i$, and for $\varepsilon > 0$ small enough,

$$p_i = \begin{cases} \varepsilon & \text{if } i \in U\backslash\{i_U\} \\ \varepsilon^2 & \text{if } i \in U'\backslash\{i_{U'}\} \\ 1-\varepsilon^2|U'\backslash\{i_{U'}\}| & \text{if } i = i_{U'} \\ \varepsilon^3 & \text{if } i \in V \\ 4 & \text{otherwise,} \end{cases}$$

and $l_{i_U} = 1-\varepsilon|U\backslash\{i_U\}|-\varepsilon^3|V|$ and $u_{i_U} = 1-\varepsilon|U\backslash\{i_U\}|$. Now, $\mathrm{AC}(N, \succeq, 1) = U' \cup \{X \mid U \subset X \subset UV\}$. Since $U\rho U'$ and $UV\rho X$ for each $X \in \mathrm{AC}(N, \succeq, 1)\backslash\{UV, U'\}$, by Claim 1, $UV\rho U'$.

Case 4 $U' = \varnothing$ and $V' \neq \varnothing$. Since the argument is symmetric to the previous case, we omit it. □

Claim 4 *Let U, V be such that $U \cap V = \varnothing$ and $U\rho V$. Then, for each $X \subset V$, $U\rho X$.*

Proof If $X = \varnothing$, then $U\rho X$ follows from property (i) of ρ. Assume $X \neq \varnothing$ and take $i_X \in X$ and $i_U \in U$. Consider any problem $(N, \succeq, 1)$ with $UV \subseteq N$ and for all $i \in N\backslash\{i_X\}$, $l_i = p_i = u_i$ and for $\varepsilon > 0$ small enough,

$$p_i = \begin{cases} \varepsilon & \text{if } i \in U\backslash\{i_U\} \\ 1-\varepsilon|U\backslash\{i_U\}| & \text{if } i = i_U \\ \varepsilon^2 & \text{if } i \in X\backslash\{i_X\} \\ \varepsilon^3 & \text{if } i \in V\backslash X \\ 4 & \text{otherwise,} \end{cases}$$

and $l_{i_X} = 1-\varepsilon^2|X\backslash\{i_X\}|-\varepsilon^3|V\backslash X|$ and $u_{i_X} = 1-\varepsilon^2|X\backslash\{i_X\}|$. Now $\mathrm{AC}(N, \succeq, 1) = \{\varnothing, U \cup \{Y \mid X \subset Y \subset V\}\}$. Since $U\rho V$ and $V\rho Y$ for each $Y \in A(N, \succeq, 1)\backslash\{V, U\}$ we conclude, by Claim 1, that $U\rho X$. □

Claim 5 *Assume that for each $X, Y \in \{A, B, C, D, E, F\}$, $X \cap Y = \varnothing$, $AB\rho DE$, and $CD\rho AF$. Then, $ABCD\rho DEAF$.*

Proof We first prove that if $B \neq \varnothing$, $D \neq \varnothing$, and $F \neq \varnothing$, then $ABCD\rho DEAF$. Let $S = ABC$, $S' = CDE$, and $S'' = AEF$. Since $AB\rho DE$, $CD\rho AF$, and ρ satisfies property (ii), $S = ABC\rho CDE = S'$ and $S' = CDE\rho AEF = S''$. By Claim 2, $S = ABC\rho AEF = S''$. By Lemma 10, $BC\rho EF$. By property (ii) of ρ, $ABCD\rho DEAF$.

We now prove that if $C \neq \varnothing$, $A \neq \varnothing$, and $E \neq \varnothing$, then $ABCD\rho DEAF$. Let $S = BCD$, $S' = ABF$, and $S'' = DEF$. Since $AB\rho DE$, $CD\rho AF$, and ρ satisfies property (ii), $S' = ABF\rho DEF = S''$ and $S = B\rho ABF = S'$. By Claim 2, $S = CDB\rho DEF = S''$. By Lemma 10, $BC\rho EF$. By property (ii) of ρ, $ABCD\rho DEAF$.

We proceed by considering several cases:

Case 1 $A = \varnothing$, $D = \varnothing$. Thus, $B\rho E$ and $C\rho F$. Then, $BC\rho EF$ follows from Claim 3 and hence $ABCD\rho DEAF$.

Case 2 $A = \varnothing$, $D \neq \varnothing$. Thus, $B\rho DE$ and $CD\rho F$. Since $B\rho DE$, $B \neq \varnothing$. We consider two subcases.

Subcase 2.1 $F \neq \varnothing$. Since $B \neq \varnothing$, $D \neq \varnothing$, and $F \neq \varnothing$, $ABCD\rho DEAF$ holds.
Subcase 2.2 $F = \varnothing$. Thus, $B\rho DE$ and $C\rho D$. By property (ii) of ρ, it is sufficient to prove that $BC\rho E$. Since $B\rho DE$, by Claim 4, $B\rho E$. Since $C\rho\varnothing$ and Claim 3 holds, $BC\rho E$. Thus, $ABCD\rho DEAF$.

Case 3 $A \neq \varnothing$, $D = \varnothing$. It is symmetric to Case 2.

Case 4 $A \neq \varnothing$, $D \neq \varnothing$. We consider three subcases.
Subcase 4.1 $B \neq \varnothing$, $F \neq \varnothing$. Since $B \neq \varnothing$, $D \neq \varnothing$, and $F \neq \varnothing$, $ABCD\rho DEAF$ holds.
Subcase 4.2 $B \neq \varnothing$, $F = \varnothing$. Thus, $AB\rho DE$ and $CD\rho A$. By property (ii) of ρ, it is sufficient to prove that $B\rho E$. First, if $E = \varnothing$ it holds trivially. Second, assume $E \neq \varnothing$ and $C \neq \varnothing$ hold. Then, and since $C \neq \varnothing$, $A \neq \varnothing$, and $E \neq \varnothing$, $ABCD\rho DEAF$ holds. Finally, assume $E \neq \varnothing$ and $C = \varnothing$ hold. Suppose $E\rho B$. By Claim 3, $DE\rho AB$, which contradicts that $AB\rho DE$.
Subcase 4.3 $B = \varnothing$. Thus, $A\rho DE$ and $CD\rho AF$. We first prove that $C \neq \varnothing$. Suppose not. Then, $D\rho AF$. By Claim 4, $D\rho A$. Since $A\rho DE$, and by Claim 4 again, $A\rho D$, which contradicts the antisymmetry of ρ. Hence, $C \neq \varnothing$. First, assume $E = \varnothing$. Thus, $A\rho D$ and $CD\rho AF$. By property (ii) of ρ, it is sufficient to prove that $C\rho F$. Suppose not. Then, $F\rho C$. Since $A\rho D$ and Claim 3, FAρCD, which contradicts that ρ is antisymmetric and $CD\rho AF$. Second, assume $E \neq \varnothing$. Since $C \neq \varnothing$, $A \neq \varnothing$, and $E \neq \varnothing$, $ABCD\rho DEAF$ holds. □

To conclude with the proof of Lemma 13, assume $S\rho S'$ and $S'\rho S''$. We want to show that $S\rho S''$ holds. Since $S\rho S'$, $ABCG\rho CDEG$ (see Fig. 2). By Lemma 10, $AB\rho DE$. Since $S'\rho S''$, $CDEG\rho AEFG$. By Lemma 10, $CD\rho AF$. By Claim 5, $ABCD\rho DEAF$. By Lemma 10, $BC\rho EF$. By property (ii) of ρ, $S = BCAG\rho EFAG = S''$. □

A2.2. The independence of the axioms

Let σ be such that $\sigma(i) = i$ for all $i \in \mathbb{N}$. Given $S, T \in \mathcal{N}$ define $\mathbf{1}^{S,S\cup T} \in \mathbb{R}^{S\cup T}$ as follows:

$$\mathbf{1}_i^{S,S\cup T} = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Define $\mathbf{1}^{T,S\cup T}$ analogously. We define the order ρ on $\mathcal{N}$. For any $S, T \in \mathcal{N}$, $S \neq T$, set $S\rho T$ if and only if $\mathbf{1}^{S,S\cup T}$ is strictly larger, according to the lexicographic order, than $\mathbf{1}^{T,S\cup T}$. Now, it is easy to see that for any problem $(N, \succeq, t)$, $c^{F^\sigma}(N, \succeq, t) \in \mathrm{AC}(N, \succeq, t)$ and $c^{F^\sigma}(N, \succeq, t)\rho S$ for all $S \in \mathrm{AC}(N, \succeq, t) \backslash c^{F^\sigma}(N, \succeq, t)$. It is not difficult to prove that, as defined in A1.3 of Appendix 1,

1. f^1 is consistent, individually rational from equal division and satisfies independence of irrelevant coalitions but it is not efficient;
2. f^2 is efficient, consistent and satisfies independence of irrelevant coalitions but it is not individually rational from equal division; and

3. f^3 is efficient and individually rational from equal division and satisfies independence of irrelevant coalitions but it is not consistent.

We define f^4 as follows. Let σ' be the order in which agent 1 is always the last and the other agents are ordered as in σ. Now, for all $(N, \succeq, t) \in \mathcal{P}$,

$$f^4(N, \succeq, t) = \begin{cases} F^{\sigma'}(N, \succeq, t) & \text{if } 1 \in N \text{ and } p_1 = 1 \\ F^{\sigma}(N, \succeq, t) & \text{otherwise.} \end{cases}$$

It is not difficult to prove that f^4 is efficient, consistent, individually rational from equal division but it does not independence of irrelevant coalitions.

Appendix 3: Proof of Theorem 3

A3.1. Proof of the characterization

($\Longleftarrow$) Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be an order. We first prove that the extended uniform rule F^{σ} is efficient, consistent, individually rational from equal division and satisfies order priority with respect to σ. We do it in Lemmata 14 and 15 below. In order to simplify the notation, assume $\sigma(i) = i$ for all $i \in \mathbb{N}$.

Lemma 14 *The extended uniform rule F^{σ} is efficient, consistent and individually rational from equal division.*

Proof By Theorem 1, it is sufficient to prove that F^{σ} satisfies (1.a) and (1.b). By its definition, F^{σ} satisfies (1.a). To show that F^{σ} also satisfies (1.b), consider any problem $(N, \succeq, t)$ and let $i \in N$ be arbitrary. For each $1 \leq j \leq n-1$, let X'^j denote the sets X^j as in the definition of F^{σ} when the procedure is applied to the problem $(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, \widehat{t})$, where

$$\widehat{t} = \begin{cases} t & \text{if } F_i^{\sigma}(N, \succeq, t) = \mathrm{NP} \\ t - F_i^{\sigma}(N, \succeq, t) & \text{otherwise.} \end{cases}$$

We will prove that

$$c^{F^{\sigma}}(N, \succeq, t)\backslash\{i\} \in X'^j \quad \text{for all } 1 \leq j \leq n-1. \tag{12}$$

Observe that (1.b) would follow because (12) and $|X'^{n-1}| = 1$ imply that $c^{F^{\sigma}}(N, \succeq, t)\backslash\{i\} = X'^{n-1}$ and hence, $c^{F^{\sigma}}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, \widehat{t}) = c^{F^{\sigma}}(N, \succeq, t)\backslash\{i\}$. To prove (12) we consider separately two cases.

Case 1 $F_i^{\sigma}(N, \succeq, t) \in [l_i, u_i]$. Thus, $i \in c^f(N, \succeq, t)$. We first mention two statements:

($s1$) Let $S \in \mathrm{AC}(N\backslash\{i\}, \succeq_{N\backslash\{i\}}, t - F_i^{\sigma}(N, \succeq, t))$. Then, $\sum_{j\in S} l_j \leq t - F_i^{\sigma}(N, \succeq, t) \leq \sum_{j\in S} u_j$. Hence, $\sum_{j\in S\cup\{i\}} l_j \leq t \leq \sum_{j\in S\cup\{i\}} u_j$. Namely, $S \cup \{i\} \in \mathrm{AC}(N, \succeq, t)$.

($s2$) Let $S \in \text{AC}(N, \succeq, t)$ be such that $i \in S$ and there exists $(x_j)_{j \in S} \in \text{FA}(S, \succeq_S, t)$ such that $x_i = F_i^\sigma(N, \succeq, t)$. Thus, $S \backslash \{i\} \in \text{AC}(N \backslash \{i\}, \succeq_{N \backslash \{i\}}, t - F_i^\sigma(N, \succeq, t))$.

Since $c^{F^\sigma}(N, \succeq, t) \in X^0 \equiv \text{AC}(N, \succeq, t)$ and ($s2$) holds,

$$c^{F^\sigma}(N, \succeq, t) \backslash \{i\} \in X'^0 = \text{AC}(N \backslash \{i\}, \succeq_{N \backslash \{i\}}, t - F_i^\sigma(N, \succeq, t)).$$

We now prove that $c^{F^\sigma}(N, \succeq, t) \backslash \{i\} \in X'^j$ for all $1 \leq j \leq n-1$. We do it for $j = 1$, the first step of the procedure (the other steps are similar and we omit them). We consider two subcases.
Subcase 1.1 For each $S \in X^0, 1 \notin S$. Then $X^1 = X^0$. Suppose that $1 \in S$ for some $S \in X'^0$. By ($s1$), $S \cup \{i\} \in X^0$, which is a contradiction. Then, for each $S \in X'^0, 1 \notin S$. Hence $X'^1 = X'^0$ and $c^{F^\sigma}(N, \succeq, t) \backslash \{i\} \in X'^1$.
Subcase 1.2 There exists $S \in X^0$ such that $1 \in S$. Then, $X^1 = \{S \in X^0 | 1 \in S\}$. Again, we consider two subcases.
Subcase 1.2.1 $i \neq 1$. Since $c^{F^\sigma}(N, \succeq, t) \in X^1$, by ($s2$), $1 \in c^{F^\sigma}(N, \succeq, t) \backslash \{i\} \in X'^0$. Now $X'^1 = \{S \in X'^0 | 1 \in S\}$ and hence $c^{F^\sigma}(N, \succeq, t) \backslash \{i\} \in X'^1$.
Subcase 1.2.2 $i = 1$. In this case we cannot compute X'^1. After X'^0 we must compute X'^2. We prove that $c^f(N, \succeq, t) \backslash \{i\} \in X'^2$. We again consider two subcases.
Subcase 1.2.2.1 For each $S \in X^1, 2 \notin S$. Then $X^2 = X^1$. Suppose that $2 \in S$ for some $S \in X'^0$. By ($s1$), $S \cup \{1\} \in X^0$, which is a contradiction. Then, for each $S \in X'^0, 2 \notin S$. Hence $X'^2 = X'^0$ and $c^{F^\sigma}(N, \succeq, t) \backslash \{1\} \in X'^2$.
Subcase 1.2.2.2 There exists $S \in X^1$ such that $2 \in S$. Then $X^2 = \{S \in X^2 | 2 \in S\}$. Since $c^{F^\sigma}(N, \succeq, t) \in X^2$, by ($s2$), $2 \in c^{F^\sigma}(N, \succeq, t) \backslash \{1\} \in X'^0$. Now $X'^2 = \{S \in X'^0 | 2 \in S\}$ and hence $c^{F^\sigma}(N, \succeq, t) \backslash \{1\} \in X'^2$.

Case 2 $F_i^\sigma(N, \succeq, t) \notin [l_i, u_i]$. Then, $F_i^\sigma(N, \succeq, t) = \text{NP}$ and $i \notin c^f(N, \succeq, t)$. It is easy to see that $\text{AC}(N \backslash \{i\}, \succeq_{N \backslash \{i\}}, t) = \{S \in \text{AC}(N, \succeq, t) \mid i \notin S\}$. Hence, $c^{F^\sigma}(N, \succeq, t) \in X'^0$. Using arguments similar to those used in Case 1, we can prove that $c^{F^\sigma}(N, \succeq, t) \in X'^j$ for all $1 \leq j \leq n-1$. □

Lemma 15 *The extended uniform rule F^σ satisfies order priority with respect to σ.*

Proof Let $i \in N$ be such that $i \notin c^{F^\sigma}(N, \succeq, t)$ and $c^{F^\sigma}(N, \succeq, t) \cap \{i+1, \ldots, n\} \neq \varnothing$. We must prove that there is no admissible coalition containing $\{1, \ldots, i\} \cap c^{F^\sigma}(N, \succeq, t)$. To obtain a contradiction, let S be an admissible coalition containing $\{1, \ldots, i\} \cap c^{F^\sigma}(N, \succeq, t)$. Let $j \in N$. If there exists $S' \in X^{j-1}$ such that $j \in S'$, then $X^j = \{T \in X^{j-1} \mid j \in T\}$. Since, $c^{F^\sigma}(N, \succeq, t) = X^n \subset X^j$, $j \in c^{F^\sigma}(N, \succeq, t)$. Thus, if $j \notin c^{F^\sigma}(N, \succeq, t)$, $\{T \in X^{j-1} \mid j \in T\} = \varnothing$ and $X^j = X^{j-1}$. We now prove that $S \in X^j$ for all $1 \leq j \leq i$. We prove it by induction. First, $S \in X^0$ holds and let $1 \leq j \leq i$. Assume that $S \in X^{j-1}$. We prove that $S \in X^j$. We distinguish between two possible cases.

Case 1 $j \notin c^{F^\sigma}(N, \succeq, t)$. Thus, $X^j = X^{j-1}$, which means that $S \in X^j$.

Case 2 $j \in c^{F^\sigma}(N, \succeq, t)$. Thus, $X^j = \{T \in X^{j-1} \mid j \in T\}$ and $S \in X^j$ because $\{1, \ldots, i\} \cap c^{F^\sigma}(N, \succeq, t) \subset S$.

Thus, $i \in S \in X^i$, which means that $i \in c^{F^\sigma}(N, \succeq, t)$. But this contradicts the initial assumption that $i \notin c^{F^\sigma}(N, \succeq, t)$. □

($\Longrightarrow$) Let f be an efficient and consistent rule that satisfies individual rationality from equal division and order priority with respect to σ. By Theorem 1, f is an extended uniform rule. Lemma 16 below finishes with the proof of the characterization in Theorem 3.

Lemma 16 *Let $(N, \succeq, t)$ be a problem. Then, $c^f(N, \succeq, t) = c^{F^\sigma}(N, \succeq, t)$.*

Proof By definition of F^σ, $c^{F^\sigma}(N, \succeq, t) = X^n$. We now prove that if f satisfies order priority with respect to σ, then $c^f(N, \succeq, t) = X^n$. We show that for each $i \in N$, $i \in c^f(N, \succeq, t)$ if and only if $i \in X^n$. Assume, without loss of generality, that $\sigma(i) = i$ for all $i \in \mathbb{N}$. We proceed by induction on the index of the agents. If there exists an admissible coalition S such that $1 \in S$, then $X^1 = \{S \in \mathrm{AC}(N, \succeq, t) \mid 1 \in S\}$. In this case $1 \in X^n$ because $X^n \subseteq X^1$. If there does not exist an admissible coalition S such that $1 \in S$, then $X^1 = \mathrm{AC}(N, \succeq, t)$. In this case, $1 \notin X^n$. Since f satisfies order priority with respect to σ, it is easy to see that $1 \in c^f(N, \succeq, t)$ if and only if there exists an admissible coalition S such that $1 \in S$.

Assume that for all $j < i \leq n$, $j \in c^f(N, \succeq, t)$ if and only if $j \in X^n$. We prove that $i \in c^f(N, \succeq, t)$ if and only if $i \in X^n$. Using arguments similar to those used with agent 1 we can prove that $i \in X^n$ if and only if there exists an admissible coalition $S \in X^{i-1}$ such that $i \in S$. We now prove that $i \in c^f(N, \succeq, t)$ if and only if there exists an admissible coalition $S \in X^{i-1}$ such that $i \in S$.

Assume $i \in c^f(N, \succeq, t)$ and let $S = c^f(N, \succeq, t)$. By definition, $c^f(N, \succeq, t)$ is admissible. By induction hypothesis, $\{1, \ldots, i-1\} \cap c^f(N, \succeq, t) = \{1, \ldots, i-1\} \cap X^n$. Thus, $c^f(N, \succeq, t) \in X^{i-1}$.

Assume that there exists an admissible coalition $S \in X^{i-1}$ such that $i \in S$. By induction hypothesis, $\{1, \ldots, i-1\} \cap c^f(N, \succeq, t) = \{1, \ldots, i-1\} \cap X^n$. Since $\{1, \ldots, i\} \cap X^n \subset S$, S is an admissible coalition containing $\{1, \ldots, i\} \cap c^f(N, \succeq, t)$. Since f satisfies order priority with respect to σ, $i \in c^f(N, \succeq, t)$. □

A3.2. The independence of the axioms

Assume, by simplicity, that $\sigma(i) = i$ for all $i \in \mathbb{N}$. We define f^5 as follows. Given $S \in \mathrm{AC}(N, \succeq, t)$, define $\mathrm{ID}_i^\sigma(S, \succeq, t)$ as the share obtained by i when agents select sequentially, following the order σ, the share they prefer most corresponding to feasible and individually rational from equal division allocations (we avoid the technical definition). Given $(N, \succeq, t)$, set $c^{f^5}(N, \succeq, t) = c^{F^\sigma}(N, \succeq, t)$ and $f_i^5(N, \succeq, t) = \mathrm{NP}$ for each $i \notin c^{f^5}(N, \succeq, t)$ and for each $i \in c^{f^5}(N, \succeq, t)$,

$$f_i^5(N, \succeq, t) = \begin{cases} F_i^\sigma(c^{F^\sigma}(N, \succeq, t), \succeq_{c^{F^\sigma}(N, \succeq, t)}, t) & \text{if } |c^{F^\sigma}(N, \succeq, t)| \text{ is odd} \\ \mathrm{ID}_i^\sigma(c^{F^\sigma}(N, \succeq, t), \succeq_{c^{F^\sigma}(N, \succeq, t)}, t) & \text{if } |c^{F^\sigma}(N, \succeq, t)| \text{ is even.} \end{cases}$$

It is not difficult to show that:

1. The rule f^1 is consistent, individually rational from equal division and satisfies order priority with respect to σ, but it is not efficient.
2. The rule f^2 is efficient, consistent and satisfies order priority with respect to σ, but it is not individually rational from equal division.
3. Any extended uniform rule $F^{\sigma'}$ with $\sigma' \neq \sigma$ is efficient, consistent and individually rational from equal division, but it does not satisfy order priority with respect to σ.
4. The rule f^5 is efficient, individually rational from equal division and satisfies order priority with respect to σ, but it is not consistent.

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